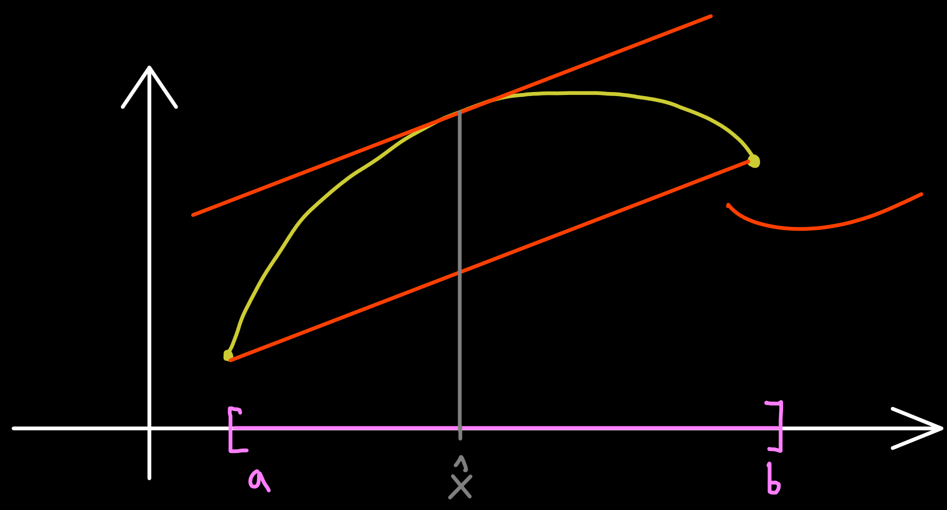
Real Analysis - Part 41



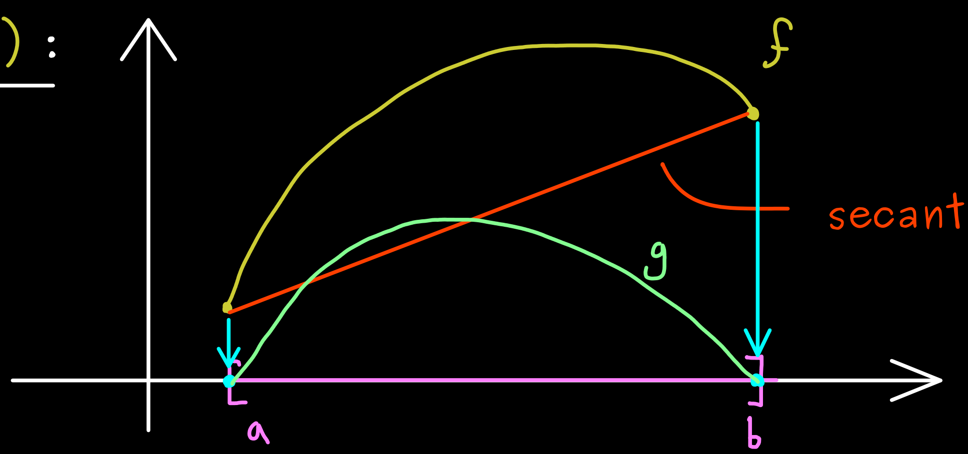
mean slope: $\dfrac{f(b)-f(a)}{b-a}$

Mean value theorem: Let $f: [a,b] \to \mathbb{R}$ be differentiable.

Then there exists $\hat{x} \in (a,b)$ with $f'(\hat{x}) = \dfrac{f(b)-f(a)}{b-a}$.

Proof: Rolle's theorem: $f(a) = f(b) \Longrightarrow$ there is $\hat{x} \in (a,b)$ with $f'(\hat{x}) = \underbrace{\dfrac{f(b)-f(a)}{b-a}}_{=0}$ ✓

If $f(a) \neq f(b)$:



Define: $g: [a,b] \to \mathbb{R}$ by $g(x) := f(x) - \left(\dfrac{f(b)-f(a)}{b-a}(x-a) + f(a) \right)$

$\Longrightarrow$ g differentiable with $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

Now: $g(a) = g(b) \overset{\text{Rolle's theorem}}{\Longrightarrow}$ there is $\hat{x} \in (a,b)$ with $g'(\hat{x}) = 0$

$$\Longrightarrow \quad f'(\hat{x}) = \frac{f(b)-f(a)}{b-a} \qquad \square$$

Application: $f : [a,b] \to \mathbb{R}$ be differentiable. Assume $f'(x) > 0$ for all $x \in [a,b]$

Then: $x_1 < x_2 \xRightarrow[f : [x_1, x_2] \to \mathbb{R}]{\text{mean value theorem}}$ there is $\hat{x} \in (x_1, x_2)$ with $f'(\hat{x}) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$\Longrightarrow \quad f(x_2) - f(x_1) = \underbrace{f'(\hat{x})}_{>0} \cdot \underbrace{(x_2 - x_1)}_{>0} > 0$$

$\Longrightarrow$ f strictly monotonically increasing

(a) $f'(x) > 0$ for all $x \in [a,b] \Longrightarrow$ f strictly monotonically increasing

(b) $f'(x) < 0$ for all $x \in [a,b] \Longrightarrow$ f strictly monotonically decreasing

(c) $f'(x) \geq 0$ for all $x \in [a,b] \Longrightarrow$ f monotonically increasing

(d) $f'(x) \leq 0$ for all $x \in [a,b] \Longrightarrow$ f monotonically decreasing