Real Analysis — Part 41

mean slope: $\dfrac{f(b) - f(a)}{b - a}$
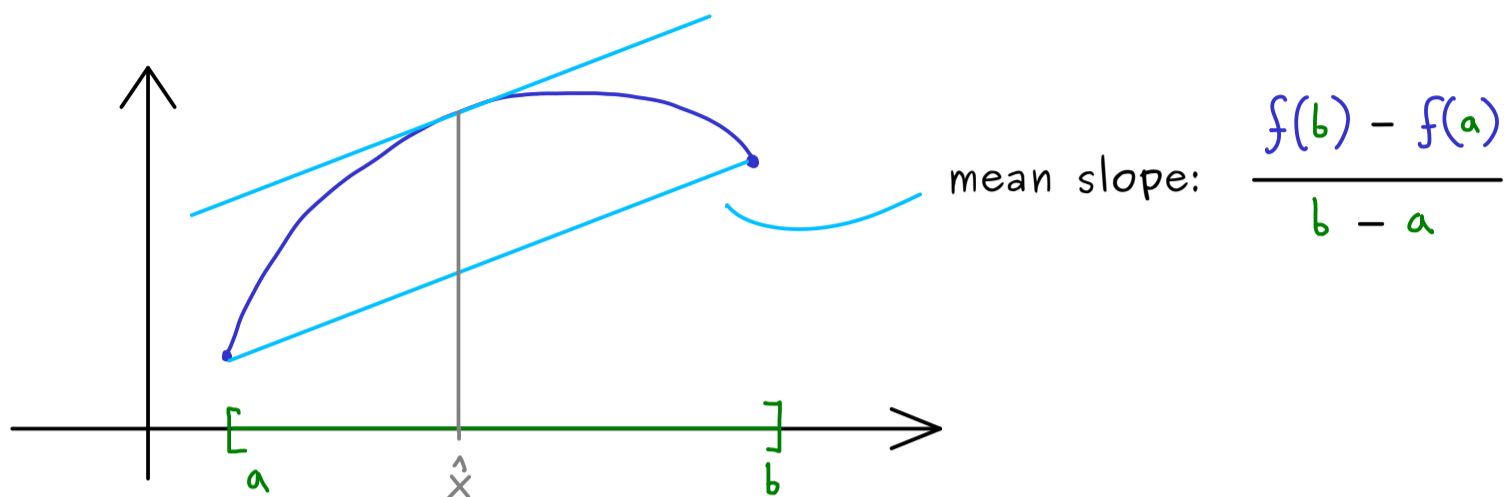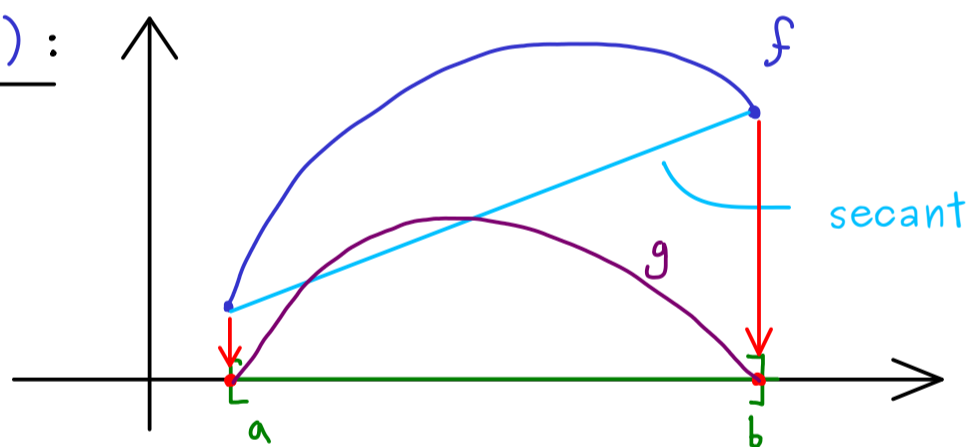
<u>Mean value theorem:</u> Let $f : [a, b] \to \mathbb{R}$ be differentiable.

Then there exists $\hat{x} \in (a, b)$ with $f'(\hat{x}) = \dfrac{f(b) - f(a)}{b - a}$.

<u>Proof:</u> Rolle's theorem: $f(a) = f(b) \implies$ there is $\hat{x} \in (a, b)$ with $f'(\hat{x}) = \dfrac{f(b) - f(a)}{b - a}$ ✓

$$\underbrace{\phantom{\dfrac{f(b) - f(a)}{b - a}}}_{= 0}$$

If $\underline{f(a) \neq f(b)}$:

secant

Define: $g : [a, b] \to \mathbb{R}$ by $g(x) := f(x) - \left(\dfrac{f(b) - f(a)}{b - a} (x - a) + f(a) \right)$

$\implies$ g differentiable with $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$

Now: $g(a) = g(b) \overset{\text{Rolle's theorem}}{\implies}$ there is $\hat{x} \in (a, b)$ with $g'(\hat{x}) = 0$

$\implies$ $f'(\hat{x}) = \dfrac{f(b) - f(a)}{b - a}$ $\qquad\qquad$ $\square$

Application: $f : [a,b] \to \mathbb{R}$ be differentiable. Assume $f'(x) > 0$ for all $x \in [a,b]$

Then: $X_1 < X_2 \overset{\text{mean value theorem}}{\underset{f : [x_1, x_2] \to \mathbb{R}}{\Longrightarrow}}$ there is $\hat{x} \in (x_1, x_2)$ with $f'(\hat{x}) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\Longrightarrow \quad f(x_2) - f(x_1) = \underbrace{f'(\hat{x})}_{>0} \cdot \underbrace{(x_2 - x_1)}_{>0} \quad > 0$

$\Longrightarrow \quad f$ strictly monotonically increasing

(a) $f'(x) > 0$ for all $x \in [a,b] \Longrightarrow f$ strictly monotonically increasing

(b) $f'(x) < 0$ for all $x \in [a,b] \Longrightarrow f$ strictly monotonically decreasing

(c) $f'(x) \geq 0$ for all $x \in [a,b] \Longrightarrow f$ monotonically increasing

(d) $f'(x) \leq 0$ for all $x \in [a,b] \Longrightarrow f$ monotonically decreasing