The Bright Side of Mathematics

Real Analysis – Part 42

Extended mean value theorem: $f, g : [a,b] \to \mathbb{R}$ be differentiable and $g'(x) \neq 0$ for all $x \in (a,b)$.

Then there exists $\hat{x} \in (a,b)$ with $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(\hat{x})}{g'(\hat{x})}$

(If $g(x) = x$, we get the normal mean value theorem)

Proof: We will use Rolle's theorem again.

Define: $h : [a,b] \to \mathbb{R}$ by $h(x) := f(x) - \left(\dfrac{f(b)-f(a)}{g(b)-g(a)} \cdot \big(g(x) - g(a)\big) + f(a) \right)$

We have: $h(a) = h(b)$ and h differentiable

$\overset{\text{Rolle's theorem}}{\Longrightarrow}$ there is $\hat{x} \in (a,b)$ with $h'(\hat{x}) = 0$

$$f'(\hat{x}) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(\hat{x}) \qquad \square$$

L'Hospital's rule: Let I be an interval and $f, g : I \to \mathbb{R}$ be differentiable. Let $x_0 \in I$ with $f(x_0) = g(x_0) = 0$ and $g'(x) \neq 0$ for $x \neq x_0$. (at least in a neighbourhood of x_0)

Then:

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists} \quad \Longrightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} \text{ exists}$$

$$\text{and} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Proof: Choose sequence $(x_n)_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}$ with $x_n \xrightarrow{n \to \infty} x_0$.

Apply extended mean value theorem for $[a,b] = [x_n, x_0]$ or $= [x_0, x_n]$

$\Longrightarrow$ there is a sequence $(\hat{x}_n)_{n \in \mathbb{N}}$ with $\hat{x}_n \in (x_n, x_0)$ or (x_0, x_n) and $\hat{x}_n \xrightarrow{n \to \infty} x_0$ satisfying:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \xleftarrow{n \to \infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(\hat{x}_n)}{g'(\hat{x}_n)} \xrightarrow{n \to \infty} \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \qquad \square$$

Example:

(a) $\displaystyle \lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$

(b) $\displaystyle \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{+\sin(x)}{2 \cdot x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$