Real Analysis - Part 47

Taylor:
$$f(x_0+h) = T_n(h) + R_n(h)$$

$$T_n(h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{n\text{-th order Taylor polynomial}} \qquad R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$$

ξ between x_0 and x_0+h

Proof:
$$F_{n,h}(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot (h+x_0-t)^k$$

Note: $F_{n,h}(x_0) = T_n(h)$

$F_{n,h}(x_0+h) = f(x_0+h)$

$$g_{n,h}(t) := (h+x_0-t)^{n+1}, \quad g'_{n,h}(t) = -(n+1)\cdot(h+x_0-t)^n$$

Generalised mean value theorem:
$$\frac{F_{n,h}(x_0+h) - F_{n,h}(x_0)}{g_{n,h}(x_0+h) - g_{n,h}(x_0)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

ξ between x_0 and x_0+h

$$f(x_0+h) - T_n(h) = \Big(\overbrace{g_{n,h}(x_0+h)}^{0} - \overbrace{g_{n,h}(x_0)}^{h^{n+1}}\Big) \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)} = \frac{h^{n+1}\cdot F'_{n,h}(\xi)}{(n+1)\cdot(h+x_0-\xi)^n}$$

$$= \frac{h^{n+1}\cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot \cancel{(h+x_0-\xi)^n}}{(n+1)\cdot\cancel{(h+x_0-\xi)^n}}$$

$$= h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\boxed{\begin{aligned} F'_{n,h}(t) &= \frac{d}{dt} \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot (h+x_0-t)^k \\ &= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!} \cdot (h+x_0-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!} \cdot (h+x_0-t)^{k-1} \\ &= \frac{f^{(n+1)}(t)}{n!} \cdot (h+x_0-t)^n \end{aligned}}$$