Real Analysis — Part 47

Taylor: $f(x_0 + h) = T_n(h) + R_n(h)$

$$\underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{\substack{n-\text{th order} \\ \text{Taylor polynomial}}} \qquad \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$$

ξ between x_0 and $x_0 + h$

Proof: $\quad F_{n,h}(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot \left(h + x_0 - t\right)^k$

Note: $\quad F_{n,h}(x_0) = T_n(h)$

$\quad F_{n,h}(x_0 + h) = f(x_0 + h)$

$$g_{n,h}(t) := \left(h + x_0 - t\right)^{n+1} \quad, \quad g'_{n,h}(t) = -(n+1) \cdot \left(h + x_0 - t\right)^n$$

Generalised mean value theorem: $\quad \dfrac{F_{n,h}(x_0 + h) - F_{n,h}(x_0)}{g_{n,h}(x_0 + h) - g_{n,h}(x_0)} = \dfrac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$

ξ between x_0 and $x_0 + h$

$$f(x_0 + h) - T_n(h) = \left(\overbrace{g_{n,h}(x_0+h)}^{0} - \overbrace{g_{n,h}(x_0)}^{h^{n+1}} \right) \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)} = \frac{h^{n+1} \cdot F'_{n,h}(\xi)}{(n+1) \cdot \left(h + x_0 - \xi\right)^n}$$

$$F'_{n,h}(t) = \frac{d}{dt} \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot \left(h + x_0 - t\right)^k$$

$$= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!} \cdot \left(h + x_0 - t\right)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!} \cdot \left(h + x_0 - t\right)^{k-1}$$

$$= \frac{f^{(n+1)}(t)}{n!} \cdot \left(h + x_0 - t\right)^n$$

$$= \frac{h^{n+1} \cdot \dfrac{f^{(n+1)}(\xi)}{n!} \cdot \cancel{\left(h + x_0 - \xi\right)^n}}{(n+1) \cdot \cancel{\left(h + x_0 - \xi\right)^n}}$$

$$= h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!}$$