Real Analysis – Part 50

Riemann integral for step function: $\int_a^b \phi(x)\, dx$

map: $\mathcal{S}([a,b]) \longrightarrow \mathbb{R}$

$\phi \longmapsto \int_a^b \phi(x)\, dx$

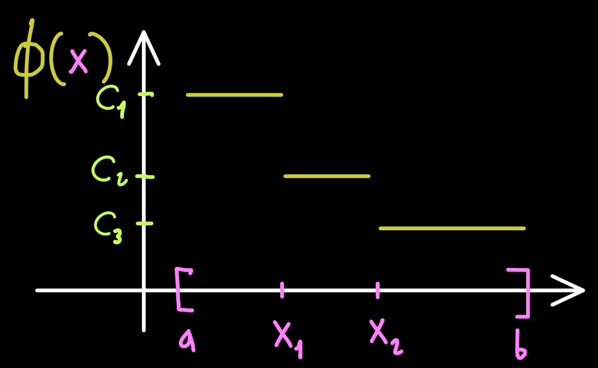
is linear and monotonic

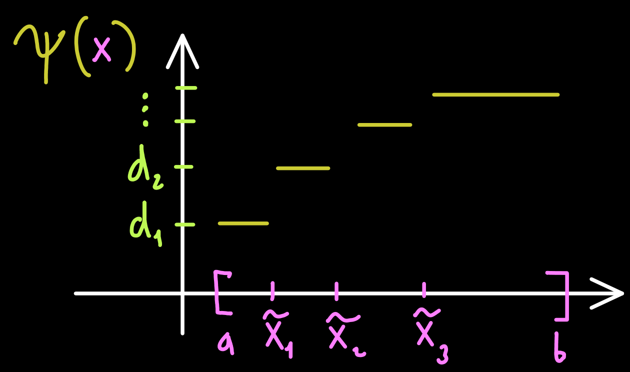
Proposition: (1) For $\lambda \in \mathbb{R}$: $\int_a^b \overbrace{\lambda \phi}^{\text{step function}}(x)\, dx = \lambda \cdot \int_a^b \phi(x)\, dx$ (homogeneous)

(2) For $\phi, \psi \in \mathcal{S}([a,b])$: $\int_a^b \overbrace{(\phi + \psi)}^{\text{step function}}(x)\, dx = \int_a^b \phi(x)\, dx + \int_a^b \psi(x)\, dx$ (additive)

(3) For $\phi, \psi \in \mathcal{S}([a,b])$: $\phi \leq \psi \Rightarrow \int_a^b \phi(x)\, dx \leq \int_a^b \psi(x)\, dx$ (monotonic)

Proof: (2)





$P_1 : a = x_0 < x_1 < \cdots < x_n = b$

$P_2 : a = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_m = b$

Define: $P_3 = P_1 \cup P_2$: $a = \tilde{\tilde{x}}_0 < \tilde{\tilde{x}}_1 < \cdots < \tilde{\tilde{x}}_N = b$

$$\int_a^b \phi(x)\, dx + \int_a^b \psi(x)\, dx = \sum_{j=1}^{N} c_j \cdot (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) + \sum_{j=1}^{N} d_j \cdot (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1})$$

$$= \sum_{j=1}^{N} (c_j + d_j)(\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) = \int_a^b (\phi + \psi)(x)\, dx$$