

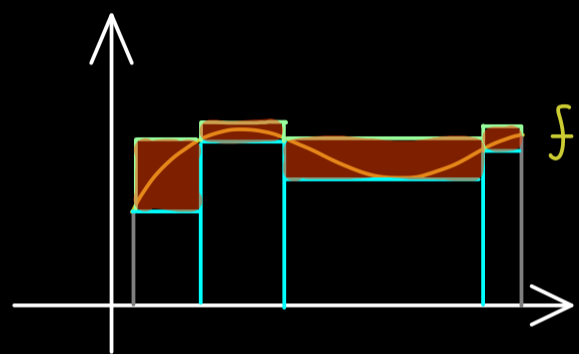
Real Analysis - Part 52

Definition: A bounded function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann-integrable

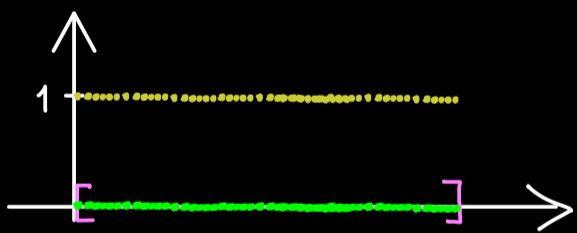
$$\text{if } \sup \left\{ \int_a^b \phi(x) dx \mid \phi \in \mathcal{S}([a, b]), \phi \leq f \right\} \\ = \inf \left\{ \int_a^b \psi(x) dx \mid \psi \in \mathcal{S}([a, b]), \psi \geq f \right\}$$

$$\Leftrightarrow \forall \varepsilon > 0 \quad \exists \phi, \psi \in \mathcal{S}([a, b]) :$$

$$\phi \leq f \leq \psi \quad \text{and} \quad \int_a^b \psi(x) dx - \int_a^b \phi(x) dx < \varepsilon$$



Examples: (a) Dirichlet function $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$



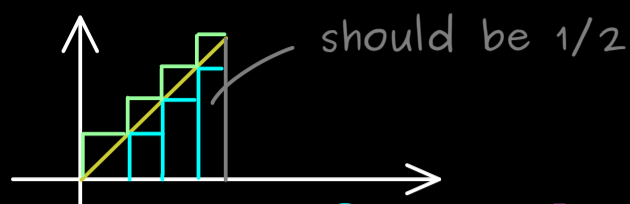
• step function ψ with $f \leq \psi$ also satisfies $1 \leq \psi$

• step function ϕ with $\phi \leq f$ also satisfies $\phi \leq 0$

$$\underbrace{\int_a^b \psi(x) dx}_{\geq 1} - \underbrace{\int_a^b \phi(x) dx}_{\leq 0} \geq 1 \Rightarrow f \text{ is not Riemann-integrable}$$

(b)

$$f: [0, 1] \rightarrow \mathbb{R}, \quad f(x) = x$$



$$\text{Define } \phi_n(x) := \frac{k-1}{n} \text{ for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \quad \phi_4(x) = \begin{cases} 0 & , x \in [0, \frac{1}{4}) \\ \frac{1}{4} & , x \in [\frac{1}{4}, \frac{2}{4}) \\ \frac{2}{4} & , x \in [\frac{2}{4}, \frac{3}{4}) \\ \frac{3}{4} & , x \in [\frac{3}{4}, \frac{4}{4}] \end{cases}$$

$$\text{Then: } \int_0^1 \phi_n(x) dx = \sum_{k=1}^n \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{k=1}^n (k-1) = \frac{1}{n^2} \cdot \frac{n \cdot (n-1)}{2} = \frac{1}{2} - \frac{1}{2n}$$

$$\text{Define } \psi_n(x) := \frac{k}{n} \text{ for } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$$

$$\text{Then: } \int_0^1 \psi_n(x) dx = \sum_{k=1}^n \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{k=1}^n k = \frac{1}{n^2} \cdot \frac{n \cdot (n+1)}{2} = \frac{1}{2} + \frac{1}{2n}$$

$\Rightarrow f$ is Riemann-integrable