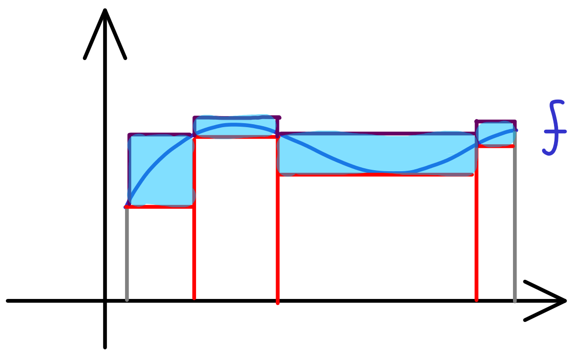
Real Analysis - Part 52

Definition: A bounded function $f: [a,b] \longrightarrow \mathbb{R}$ is Riemann-integrable

if $\sup \left\{ \int_a^b \phi(x)\,dx \;\middle|\; \phi \in S([a,b]),\ \phi \leq f \right\}$

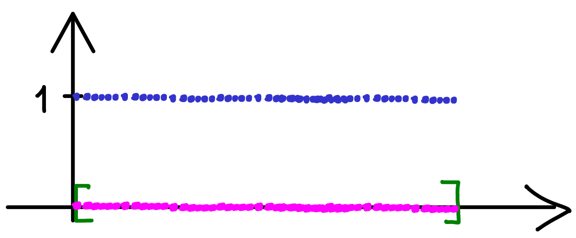
$= \inf \left\{ \int_a^b \phi(x)\,dx \;\middle|\; \phi \in S([a,b]),\ \phi \geq f \right\}$



$$\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists \phi, \psi \in S([a,b]):$$

$$\phi \leq f \leq \psi \quad \text{and} \quad \int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \varepsilon$$

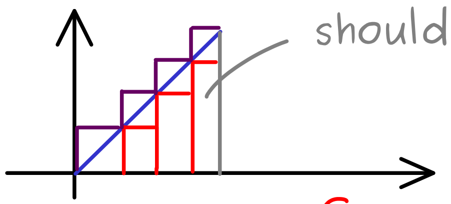
Examples: (a) Dirichlet function $f: [0,1] \longrightarrow \mathbb{R}$, $f(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$



- step function ψ with $f \leq \psi$ also satisfies $1 \leq \psi$
- step function ϕ with $\phi \leq f$ also satisfies $\phi \leq 0$

$$\underbrace{\int_a^b \psi(x)\,dx}_{\geq 1} - \underbrace{\int_a^b \phi(x)\,dx}_{\leq 0} \geq 1 \quad \Longrightarrow \quad f \text{ is not Riemann-integrable}$$

(b) $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = x$



Define $\phi_n(x) := \frac{k-1}{n}$ for $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$

$$\phi_4(x) = \begin{cases} 0 , & x \in [0, \frac{1}{4}) \\ 1/4 , & x \in [\frac{1}{4}, \frac{2}{4}) \\ 2/4 , & x \in [\frac{2}{4}, \frac{3}{4}) \\ 3/4 , & x \in [\frac{3}{4}, \frac{4}{4}] \end{cases}$$

Then: $\displaystyle\int_0^1 \phi_n(x)\, dx = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{k=1}^{n} (k-1) = \frac{1}{n^2} \cdot \frac{n \cdot (n-1)}{2} = \frac{1}{2} - \frac{1}{2n}$

Define $\psi_n(x) := \frac{k}{n}$ for $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$

Then: $\displaystyle\int_0^1 \psi_n(x)\, dx = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{k=1}^{n} k = \frac{1}{n^2} \cdot \frac{n \cdot (n+1)}{2} = \frac{1}{2} + \frac{1}{2n}$

$\Longrightarrow f$ is Riemann-integrable