Real Analysis - Part 57

Integration by substitution

$I \subseteq \mathbb{R}$ interval, $f: I \to \mathbb{R}$ continuous, $\phi: [a,b] \to I$ continuously differentiable

Then:
$$\int_a^b f(\underbrace{\phi(t)}_{=x}) \cdot \underbrace{\phi'(t)\, dt}_{dx} = \int_{\phi(a)}^{\phi(b)} f(x)\, dx$$

Remember: $x = \phi(t)$

$$\frac{dx}{dt} = \phi'(t) \Rightarrow dx = \phi'(t)\, dt$$

Example:
$$\int_0^1 t^2 \cdot \sin(t^3)\, dt = \frac{1}{3} \int_0^1 \sin(t^3) \cdot 3t^2\, dt = \frac{1}{3} \int_0^1 \sin(x)\, dx$$

$$\boxed{\begin{aligned} x &= t^3 \\ dx &= 3t^2\, dt \end{aligned}}$$

Proof: Let $F: I \to \mathbb{R}$ be an antiderivative of f

$$(F \circ \phi)'(t) \overset{\text{chain rule}}{=} F'(\phi(t)) \cdot \phi'(t) = f(\phi(t)) \cdot \phi'(t)$$

$$\int_a^b f(\phi(t)) \cdot \phi'(t)\, dt = \int_a^b (F \circ \phi)'(t)\, dt = (F \circ \phi)(t) \Big|_{t=a}^{t=b}$$

$$= F(x) \Big|_{x=\phi(a)}^{x=\phi(b)} = \int_{\phi(a)}^{\phi(b)} f(x)\, dx \qquad \square$$

Another substitution rule: $f:[a,b]\longrightarrow\mathbb{R}$ continuous, $\phi: J\longrightarrow I$ continuously differentiable and bijective

$J, I\subseteq\mathbb{R}$ intervals, $I\supseteq[a,b]$

$$\int_a^b f(x)\,dx = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(t))\cdot\phi'(t)\,dt$$

Example: $\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx$ $\quad b\in[0,1)$

try: $\phi(t) = \sin(t)$ — bijective: $[0,\frac{\pi}{2}]\longrightarrow[0,1]$

substitution: $x = \sin(t)$

$dx = \cos(t)\,dt$

$\sin^2(t)+\cos^2(t)=1$

$$= \int_0^{\arcsin(b)} \frac{1}{\underbrace{\sqrt{1-(\sin(t))^2}}_{\cos(t)}}\cos(t)\,dt = \int_0^{\arcsin(b)} 1\,dt = \arcsin(b)$$

try: $\phi(t) = \sin(t)$

substitution: $x = \sin(t)$

$dx = \cos(t)\,dt$