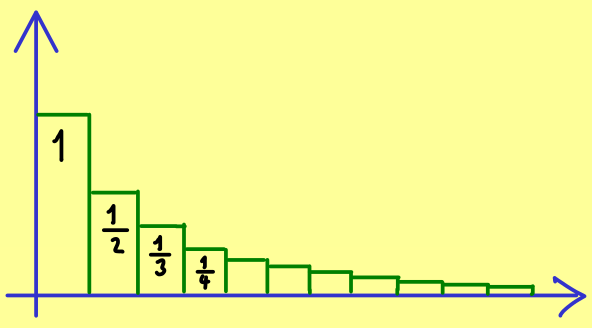
The Bright Side of Mathematics

Real Analysis – Part 18

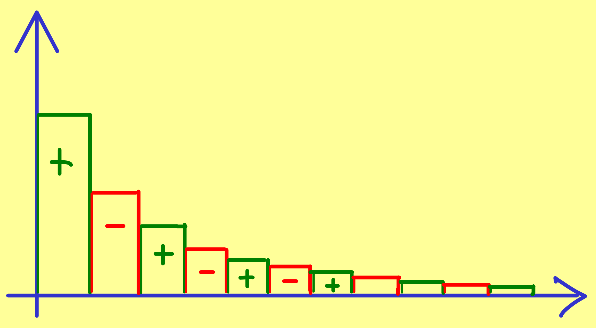
Harmonic series:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

divergent



Leibniz criterion:



$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$$

convergent

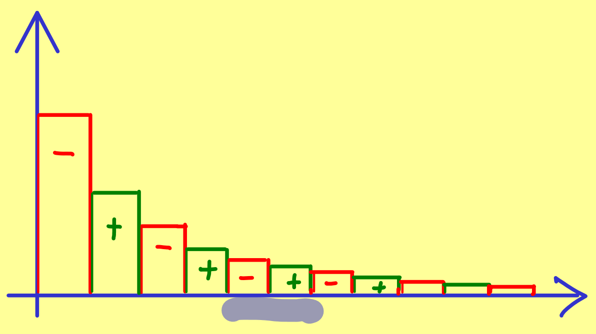
Theorem: (Alternating series test, Leibniz criterion, Leibniz's test)

Let $(a_k)_{k\in\mathbb{N}}$ be convergent with $\lim_{k\to\infty} a_k = 0$ and monotonically decreasing.

Then: $\sum_{k=1}^{\infty} (-1)^k a_k$ is convergent.

Proof: $S_n = \sum_{k=1}^{n} (-1)^k a_k$

$\Rightarrow a_k \geq 0$

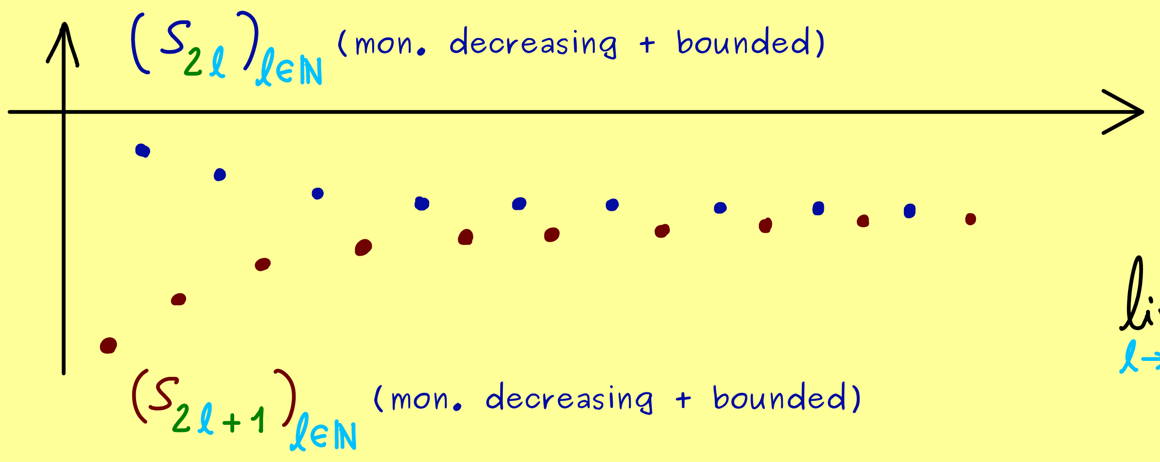


$$S_{2\ell+2} - S_{2\ell} = -a_{2\ell+1} + a_{2\ell+2} \leq 0 \qquad \text{(monotonically decreasing)}$$

$$S_{2\ell+3} - S_{2\ell+1} = a_{2\ell+2} - a_{2\ell+3} \geq 0 \qquad \text{(monotonically increasing)}$$

$$S_{2\ell+1} - S_{2\ell} = -a_{2\ell+1} \leq 0 \quad \Rightarrow \quad S_3 \leq S_{2\ell+1} \leq S_{2\ell} \leq S_2$$

(bounded)



$$\lim_{\ell\to\infty} (S_{2\ell+1} - S_{2\ell}) = \lim_{\ell\to\infty} (-a_{2\ell+1}) = 0$$

$$S := \lim_{\ell\to\infty} S_{2\ell+1} = \lim_{\ell\to\infty} S_{2\ell} \quad \Rightarrow \quad \lim_{n\to\infty} S_n = S \qquad \text{(convergent!)}$$

Example: $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ convergent by Leibniz criterion