The Bright Side of Mathematics

Real Analysis – Part 21

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n \qquad (n \text{ is even})$$
$$= a_2 + a_1 + a_4 + a_3 + \cdots + a_n + a_{n-1}$$

Reordering does not change a finite sum!

Example: $\sum_{k=0}^{\infty} (-1)^k = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$

- is not convergent
- but has two accumulation values $0, 1$

a reordering $= 1 + 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$

- is not convergent
- but has two accumulation values $1, 2$

Example: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ convergent by the Leibniz criterion

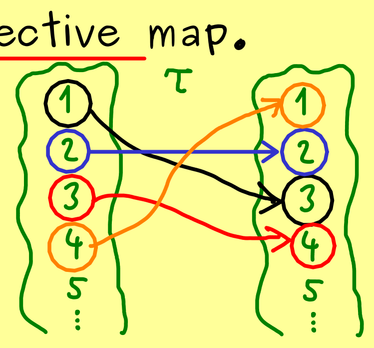
$$= \frac{1}{1} + \left(-\frac{1}{2}\right) + \frac{1}{3} + \left(-\frac{1}{4}\right) + \frac{1}{5} + \left(-\frac{1}{6}\right) + \frac{1}{7} + \cdots = \overset{= \log(2)}{C} > 0$$

a reordering $= \frac{1}{1} + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \frac{1}{9} + \frac{1}{11} + \left(-\frac{1}{6}\right) + \cdots$

$= \frac{3}{2} \cdot C$ different limits!

Definition: Let $\sum_{k=1}^{\infty} a_k$ be a series and $\tau : \mathbb{N} \to \mathbb{N}$ be a bijective map.

Then $\sum_{k=1}^{\infty} a_{\tau(k)}$ is called a reordering of $\sum_{k=1}^{\infty} a_k$.

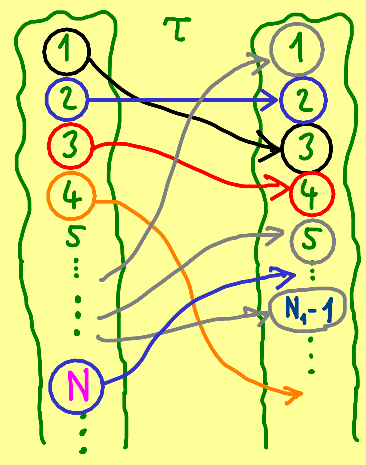


Theorem: If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then: (for any $\tau : \mathbb{N} \to \mathbb{N}$ bijective)

$\sum_{k=1}^{\infty} a_{\tau(k)}$ is also abs. convergent and $\sum_{k=1}^{\infty} a_{\tau(k)} = \sum_{k=1}^{\infty} a_k$.

Proof: Let $\varepsilon > 0$. Cauchy criterion $\Rightarrow$ $\exists N_1 \in \mathbb{N}\ \forall n \geq m \geq N_1$: $\sum_{k=m}^{n} |a_k| < \varepsilon$

$$\left| \overbrace{\sum_{k=1}^{\infty} a_k}^{A} - \sum_{k=1}^{n} a_{\tau(k)} \right| = \left| A - \sum_{k=1}^{N_1-1} a_k + \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^{n} a_{\tau(k)} \right|$$
$$\leq \left| A - \sum_{k=1}^{N_1-1} a_k \right| + \underbrace{\left| \sum_{k=1}^{N_1-1} a_k - \sum_{k=1}^{n} a_{\tau(k)} \right|}_{= (*)} < \underbrace{2 \cdot \varepsilon}_{\varepsilon'}$$
$$\left| A - \sum_{k=1}^{N_1-1} a_k \right| = \left| \sum_{k=N_1}^{\infty} a_k \right| = \lim_{n \to \infty} \left| \sum_{k=N_1}^{n} a_k \right|$$
$$\leq \lim_{n \to \infty} \sum_{k=N_1}^{n} |a_k| < \varepsilon$$



For $n \geq N$: $\{\tau(1), \tau(2), \ldots, \tau(n)\} \supseteq \{1, 2, \ldots, N_1 - 1\}$

$$\Rightarrow (*) = \left| \sum_{k=?}^{?} a_{\tau(k)} \right| \leq \sum_{k=?}^{?} |a_{\tau(k)}| \leq \sum_{j=N_1}^{\infty} |a_j| < \varepsilon$$

$$\Rightarrow \forall \varepsilon' > 0\ \exists N \in \mathbb{N}\ \forall n \geq N: \left| \overbrace{\sum_{k=1}^{\infty} a_k}^{A} - \sum_{k=1}^{n} a_{\tau(k)} \right| < \varepsilon'$$