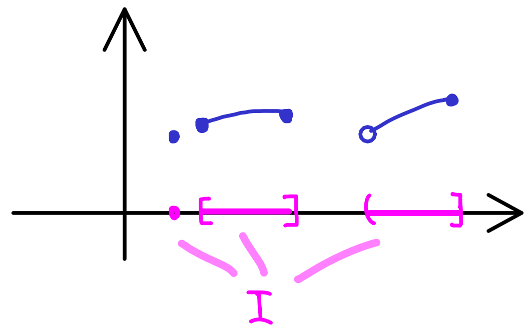
Real Analysis – Part 28

Continuity: f is called continuous at $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Theorem: Let $f: I \to \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

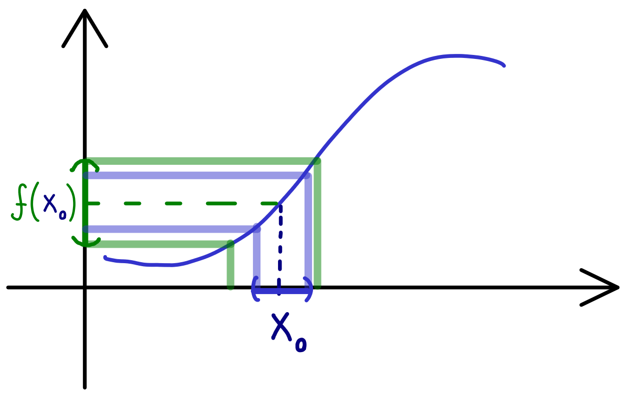
For $x_0 \in I$, we have:

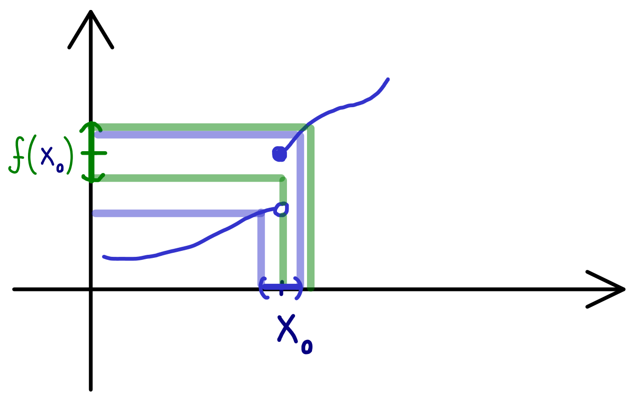


f is continuous at $x_0 \in I$

$$\Longleftrightarrow$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I: \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$





Proof: $(\Rightarrow)$ Assume $\exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \in I: \quad |x - x_0| < \delta \ \wedge \ |f(x) - f(x_0)| \geq \varepsilon$

Take $\frac{1}{n}$, $n \in \mathbb{N}$

$\implies$ For all $n \in \mathbb{N}$, we find $x_n \in I \setminus \{x_0\}$

with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$ $\implies$ f is not continuous at $x_0 \in I$

$(\Leftarrow)$ Choose sequence $(x_n)_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}$ with limit x_0. Let $\varepsilon > 0$. Take $\delta > 0$.
(from assumption)

There is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - x_0| < \delta$.

Also (by assumption) we have $|f(x_n) - f(x_0)| < \varepsilon$. $\implies$ f is continuous at $x_0 \in I$

$\square$