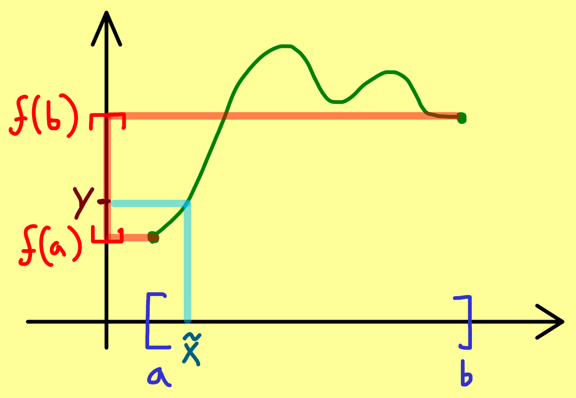
# The Bright Side of Mathematics

## Real Analysis – Part 32

$f: \underset{[a,b]}{\underset{\parallel}{I}} \longrightarrow \mathbb{R}$ continuous



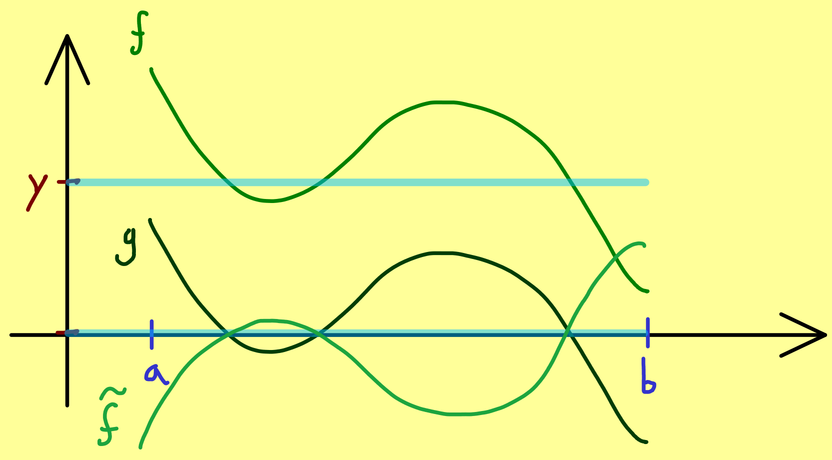

**Intermediate value theorem:** Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and

$y \in [f(a), f(b)]$ or $y \in [f(b), f(a)]$.

Then there is $\tilde{x} \in [a,b]$ with $f(\tilde{x}) = y$.

**Corollary:** $f\big[[a,b]\big]$ is also an interval.

**Proof of the intermediate value theorem:**

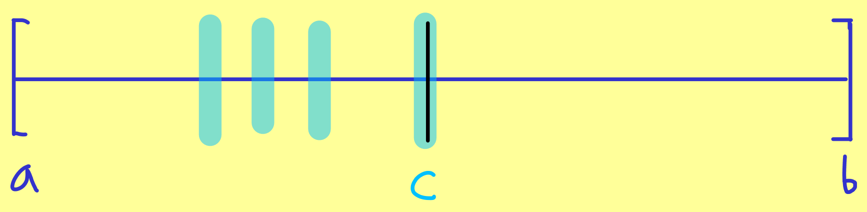



Define new function:

$g := f - y$

$$\tilde{f} := \begin{cases} -g & \text{if } g(a) > 0 \\ g & \text{if } g(a) \leq 0 \end{cases}$$

Then $\tilde{f}$ is continuous, $\tilde{y} := 0$, and $\tilde{f}(a) \leq 0$, $\tilde{f}(b) \geq 0$.




Repeat

Check $\tilde{f}(c)$:

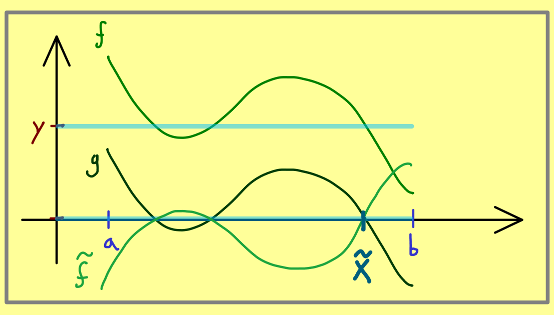
If $\tilde{f}(c) > 0$ : $b_1 := c$, $a_1 := a$

If $\tilde{f}(c) \leq 0$ : $a_1 := c$, $b_1 := b$

Wet get two Cauchy sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $b_n - a_n \xrightarrow{n\to\infty} 0$

$$\Longrightarrow \tilde{x} := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a,b]$$

We know: $\lim_{n\to\infty} \tilde{f}(a_n) \leq 0$, $\lim_{n\to\infty} \tilde{f}(b_n) \geq 0$ $\Longrightarrow$ $\tilde{f}\left(\lim_{n\to\infty} a_n\right) \leq 0$, $\tilde{f}\left(\lim_{n\to\infty} b_n\right) \geq 0$ $\Longrightarrow$ $\tilde{f}(\tilde{x}) \leq 0$, $\tilde{f}(\tilde{x}) \geq 0$




$$\Longrightarrow \tilde{f}(\tilde{x}) = 0 \Longrightarrow \underset{f(\tilde{x}) - y}{\underset{\parallel}{g(\tilde{x})}} = 0 \Longrightarrow f(\tilde{x}) = y \qquad \square$$