The Bright Side of Mathematics

Real Analysis – Part 35

f differentiable at x_0 $\iff$ $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists (call it $f'(x_0)$)

$\iff$ $\Delta_{f,x_0}(x) := \frac{f(x) - f(x_0)}{x - x_0}$ for $x \neq x_0$

can be extended to a function that is continuous at x_0

$\Delta_{f,x_0} : I \longrightarrow \mathbb{R}$ with $\lim\limits_{x \to x_0} \Delta_{f,x_0}(x) = \Delta_{f,x_0}(x_0)$

$\iff$ There is $\Delta_{f,x_0} : I \longrightarrow \mathbb{R}$ with

$$f(x) = f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x) \quad \text{for all } x \in I$$

and Δ_{f,x_0} is continuous at x_0.

$\boxed{\Delta_{f,x_0}(x) = f'(x_0) + r(x)}$

$\iff$ There is $r : I \longrightarrow \mathbb{R}$ and number $b \in \mathbb{R}$ with

$$f(x) = f(x_0) + (x - x_0) \cdot \overset{f'(x_0)}{b} + (x - x_0) \cdot r(x) \quad \text{for all } x \in I$$

and r is continuous at x_0 with $r(x_0) = 0$

Proposition: f differentiable at x_0 $\implies$ f continuous at x_0

Proof: There is $\Delta_{f,x_0} : I \longrightarrow \mathbb{R}$ which is continuous at x_0.

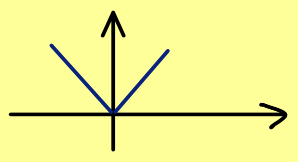
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(f(x_0) + (x - x_0) \cdot \Delta_{f,x_0}(x) \right)$$

$$= f(x_0) + \lim_{x \to x_0} (x - x_0) \cdot \lim_{x \to x_0} \Delta_{f,x_0}(x) = f(x_0) \qquad \square$$

Examples: (a) linear polynomial: $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = a_1 \cdot x + a_0$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{a_1 \cdot x + a_0 - (a_1 \cdot x_0 + a_0)}{x - x_0} = \lim_{x \to x_0} \frac{a_1 \cdot \cancel{(x - x_0)}}{\cancel{x - x_0}} = a_1$$

(b) absolute value $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = |x|$, $x_0 = 0$



$$\left.\begin{aligned} \lim_{x \searrow 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \searrow 0} \frac{x}{x} = 1 \\ &\neq \\ \lim_{x \nearrow 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \nearrow 0} \frac{-x}{x} = -1 \end{aligned}\right\} \implies f \text{ is not differentiable at } 0$$

Proposition: $f : I \longrightarrow \mathbb{R}$, $g : I \longrightarrow \mathbb{R}$ differentiable at x_0. Then:

(a) $f + g : I \longrightarrow \mathbb{R}$ differentiable at x_0 with $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

(b) $f \cdot g : I \longrightarrow \mathbb{R}$ differentiable at x_0 with $(f \cdot g)'(x_0) = f(x_0) \cdot g'(x_0) + f'(x_0) \cdot g(x_0)$

Proof for (b): $(f \cdot g)(x) = f(x) \cdot g(x) = \left(f(x_0) + (x - x_0) \Delta_{f,x_0}(x) \right) \cdot \left(g(x_0) + (x - x_0) \Delta_{g,x_0}(x) \right)$

$$= f(x_0) \cdot g(x_0) + (x - x_0) \cdot \Big(\underbrace{f(x_0) \Delta_{g,x_0}(x) + \Delta_{f,x_0}(x)\, g(x_0) + (x - x_0) \Delta_{f,x_0}(x) \Delta_{g,x_0}(x)}_{\Delta_{f \cdot g, x_0}(x) \text{ continuous}} \Big)$$

$$(f \cdot g)'(x_0) = f(x_0) \cdot g'(x_0) + f'(x_0) \cdot g(x_0)$$