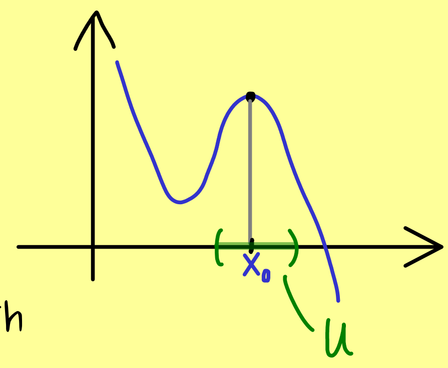
The Bright Side of Mathematics

Real Analysis – Part 40



Definition: $I \subseteq \mathbb{R}$ interval, $f: I \to \mathbb{R}$.

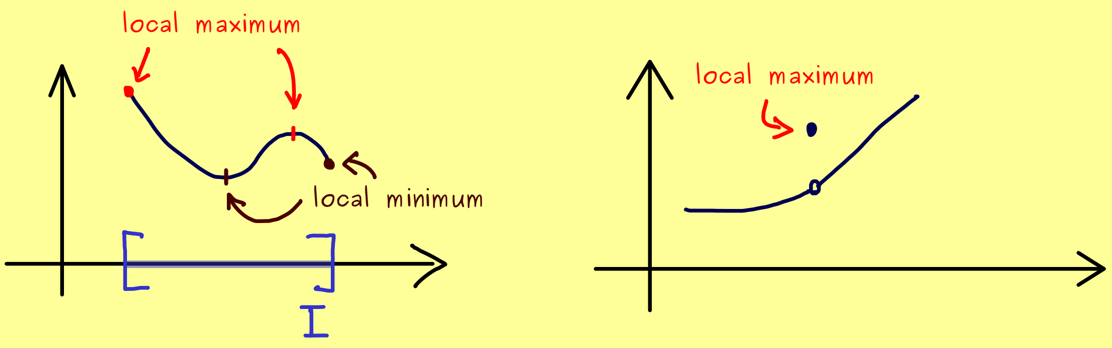
(a) f has a local maximum at $x_0 \in I$ if there is a neighbourhood of x_0, $U \subseteq \mathbb{R}$, with

$$f(x_0) = \max\{ f(x) \mid x \in U \cap I \}$$

(b) f has a local minimum at $x_0 \in I$ if there is a neighbourhood of x_0, $U \subseteq \mathbb{R}$, with

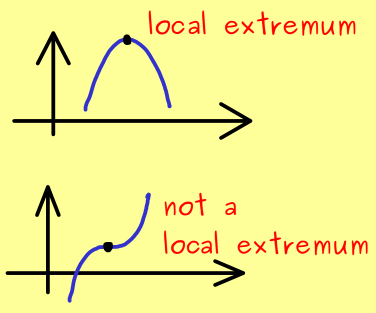
$$f(x_0) = \min\{ f(x) \mid x \in U \cap I \}$$

(c) f has a local extremum at $x_0 \in I$ if f has a local maximum or local minimum at $x_0 \in I$.



Proposition: $f: (a,b) \to \mathbb{R}$ differentiable at $x_0 \in (a,b)$.

$$f \text{ has a local extremum at } x_0 \implies f'(x_0) = 0$$

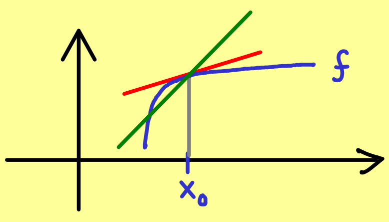


Proof: 1st case: f has a local maximum at x_0

$\implies$ there is a neighbourhood of x_0, $U \subseteq (a,b)$

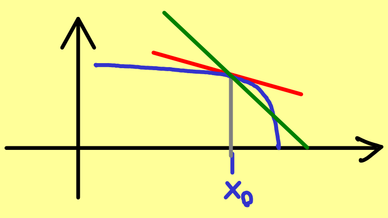
$$f(x_0) = \max\{ f(x) \mid x \in U \}$$

f differentiable at $x_0 \implies f(x) = f(x_0) + (x - x_0) \cdot \underbrace{\Delta_{f,x_0}(x)}_{\text{continuous at } x_0}$



Assume $f'(x_0) > 0$: There exists a neighbourhood $V \subseteq U$ such that $\Delta_{f,x_0}(x) > 0$ for all $x \in V$.

Then: $x > x_0 \implies f(x) = f(x_0) + \underbrace{(x - x_0)}_{>0} \cdot \underbrace{\Delta_{f,x_0}(x)}_{>0} > f(x_0)$ ↯



Assume $f'(x_0) < 0$: There exists a neighbourhood $V \subseteq U$ such that $\Delta_{f,x_0}(x) < 0$ for all $x \in V$.

Then: $x < x_0 \implies f(x) = f(x_0) + \underbrace{(x - x_0)}_{<0} \cdot \underbrace{\Delta_{f,x_0}(x)}_{<0} > f(x_0)$ ↯

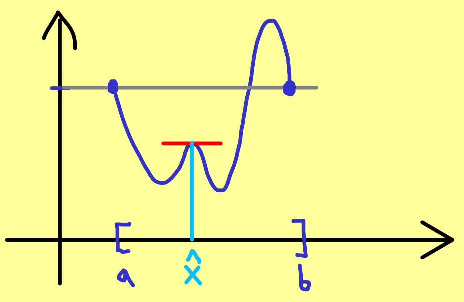
$$\implies f'(x_0) = 0$$

2nd case: f has a local minimum at x_0 (works similarly)

Theorem of Rolle

$f: [a,b] \to \mathbb{R}$ differentiable and $f(a) = f(b)$.

Then there is $\hat{x} \in (a,b)$ with $f'(\hat{x}) = 0$.



Proof: 1st case: f constant $\implies f'(x) = 0$ for all $x \in [a,b]$. ✓

2nd case: f is not constant.

There are $x^-, x^+ \in [a,b]$ with

$$f(x^+) = \sup\{ f(x) \mid x \in [a,b] \}$$
$$f(x^-) = \inf\{ f(x) \mid x \in [a,b] \}$$

$\overset{f \text{ not constant}}{\implies}$ $x^- \in (a,b)$ or $x^+ \in (a,b)$ (call it $\hat{x}$)

$\overset{\text{Proposition above}}{\implies}$ $f'(\hat{x}) = 0$ $\square$