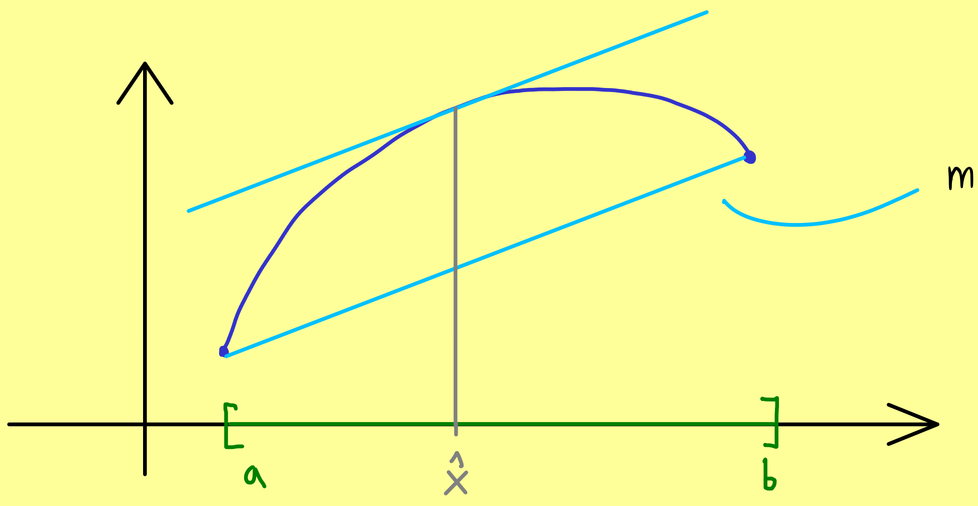
The Bright Side of Mathematics

Real Analysis – Part 41



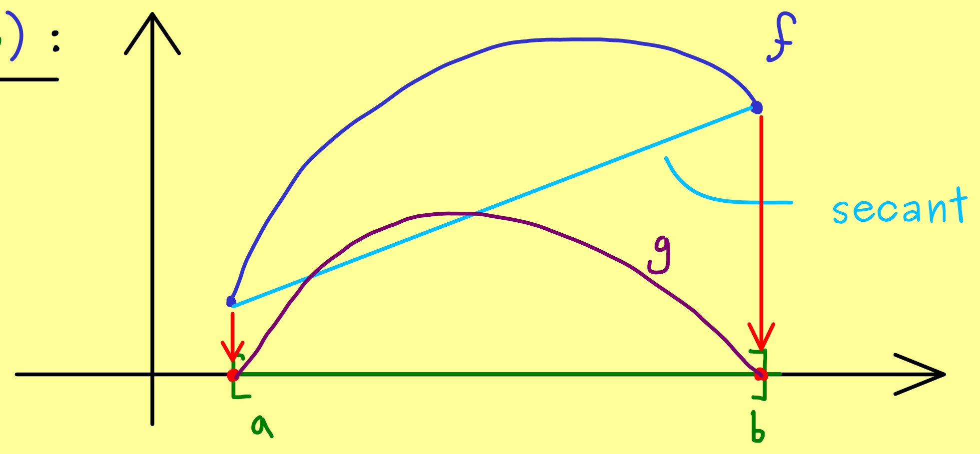
mean slope: $\dfrac{f(b)-f(a)}{b-a}$

Mean value theorem: Let $f:[a,b]\to\mathbb{R}$ be differentiable.

Then there exists $\hat{x}\in(a,b)$ with $f'(\hat{x}) = \dfrac{f(b)-f(a)}{b-a}$.

Proof: Rolle's theorem: $f(a)=f(b) \implies$ there is $\hat{x}\in(a,b)$ with $f'(\hat{x}) = \underbrace{\dfrac{f(b)-f(a)}{b-a}}_{=0}$ ✓

If $f(a)\neq f(b)$:



Define: $g:[a,b]\to\mathbb{R}$ by $g(x) := f(x) - \left(\dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$

$\implies$ g differentiable with $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

Now: $g(a) = g(b) \overset{\text{Rolle's theorem}}{\implies}$ there is $\hat{x}\in(a,b)$ with $g'(\hat{x}) = 0$

$$\implies f'(\hat{x}) = \frac{f(b)-f(a)}{b-a} \qquad \square$$

Application: $f:[a,b]\to\mathbb{R}$ be differentiable. Assume $f'(x)>0$ for all $x\in[a,b]$

Then: $x_1 < x_2 \xRightarrow[f:[x_1,x_2]\to\mathbb{R}]{\text{mean value theorem}}$ there is $\hat{x}\in(x_1,x_2)$ with $f'(\hat{x}) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

$$\implies f(x_2)-f(x_1) = \underbrace{f'(\hat{x})}_{>0}\cdot\underbrace{(x_2-x_1)}_{>0} > 0$$

$\implies$ f strictly monotonically increasing

(a) $f'(x)>0$ for all $x\in[a,b]$ $\implies$ f strictly monotonically increasing

(b) $f'(x)<0$ for all $x\in[a,b]$ $\implies$ f strictly monotonically decreasing

(c) $f'(x)\geq 0$ for all $x\in[a,b]$ $\implies$ f monotonically increasing

(d) $f'(x)\leq 0$ for all $x\in[a,b]$ $\implies$ f monotonically decreasing