BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Real Analysis – Part 43

Generalisations of l'Hospital's rule

case "$\frac{0}{0}$"

(a) I interval, $f, g : I \to \mathbb{R}$ differentiable, $x_0 \in I$, $f(x_0) = g(x_0) = 0$, $g'(x) \neq 0$ for $x \neq x_0$. Then:

$$\boxed{\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

case "$\frac{\infty}{\infty}$"

(b) I interval, $x_0 \in I$, $f, g : I \setminus \{x_0\} \to \mathbb{R}$ differentiable, $\lim_{x \to x_0} f(x) = \infty$, $\lim_{x \to x_0} g(x) = \infty$. Then:

$$\boxed{\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

case "$\frac{0}{0}$" $x \to \infty$

(c) I interval (with no upper bound), $f, g : I \to \mathbb{R}$ differentiable, $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to \infty} g(x) = 0$. Then:

$$\boxed{\lim_{x \to \infty} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to \infty} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}}$$

case "$\frac{\infty}{\infty}$" $x \to \infty$

(d) I interval (with no upper bound), $f, g : I \to \mathbb{R}$ differentiable, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to \infty} g(x) = \infty$. Then:

$$\boxed{\lim_{x \to \infty} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to \infty} \frac{f(x)}{g(x)} \text{ exists and } \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}}$$

Proof: (b) Use: $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ $\left(\frac{d}{dx} x^{-1} = (-1) \cdot x^{-2}\right)$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$

Define: $\tilde{f}(x) := \begin{cases} \frac{1}{f(x)}, & \text{for } x \in I \setminus \{x_0\} \\ 0, & \text{for } x = x_0 \end{cases}$

$\tilde{g}(x) := \begin{cases} \frac{1}{g(x)}, & \text{for } x \in I \setminus \{x_0\} \\ 0, & \text{for } x = x_0 \end{cases}$

(redo proof of l'Hospital's theorem)

$$\frac{\frac{1}{f(x_n)}}{\frac{1}{g(x_n)}} = \frac{\tilde{f}(x_n) - \tilde{f}(x_0)}{\tilde{g}(x_n) - \tilde{g}(x_0)} = \frac{\tilde{f}'(\hat{x}_n)}{\tilde{g}'(\hat{x}_n)} = \frac{\frac{f'(\hat{x}_n)}{(f(\hat{x}_n))^2}}{\frac{g'(\hat{x}_n)}{(g(\hat{x}_n))^2}}$$

(c) Define: $\tilde{f}(x) := \begin{cases} f\left(\frac{1}{x}\right), & \text{for } x > 0,\ x^{-1} \in I \\ 0, & \text{for } x = 0 \end{cases}$

Examples: (1)

$$\lim_{x \to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x} \right) = \lim_{x \to 0} \left(\frac{x - \sin(x)}{x \cdot \sin(x)} \right) \overset{\text{case (a)}}{=} \lim_{x \to 0} \left(\frac{1 - \cos(x)}{\sin(x) + x \cdot \cos(x)} \right)$$

$$= \lim_{x \to 0} \left(\frac{\sin(x)}{\cos(x) + \cos(x) - x \cdot \sin(x)} \right) = 0$$

(2)

$$\lim_{x \to \infty} \left(\frac{x}{\exp(x)} \right) \overset{\text{case (d)}}{=} \lim_{x \to \infty} \left(\frac{1}{\exp(x)} \right) = 0$$