BECOME A MEMBER
ON STEADY

The Bright Side of Mathematics

Real Analysis – Part 44

$f: I \to \mathbb{R}$ differentiable $\rightsquigarrow$ $f': I \to \mathbb{R}$

- If $f': I \to \mathbb{R}$ is continuous $\rightsquigarrow$ f continuously differentiable
- If $f': I \to \mathbb{R}$ is differentiable $\rightsquigarrow$ f two-times differentiable

$$f^{(2)} := f'' := (f')' : I \to \mathbb{R}$$

Definition: $f: I \to \mathbb{R}$ and set $f^{(0)} := f$. For $n \in \mathbb{N}$, define $f^{(n)} := \left(f^{(n-1)}\right)'$ (inductively)

- f is called n-times differentiable if $f^{(n)}$ exists.
- f is called n-times continuously differentiable if $f^{(n)}$ exists and is continuous.

(Other notations: $f^{(n)} = \frac{d^n f}{dx^n} = \frac{d^n}{dx^n} f$)

- f is called ∞-times differentiable if $f^{(n)}$ exists for all $n \in \mathbb{N}$.
  (arbitrarily often differentiable)

$$C(I) := \{ f: I \to \mathbb{R} \mid f \text{ continuous} \}$$

$$C^n(I) := \{ f: I \to \mathbb{R} \mid f \text{ n-times continuously differentiable} \}, \quad n \in \mathbb{N} \cup \{\infty\}$$

$$C(I) \supseteq C^1(I) \supseteq C^2(I) \supseteq C^3(I) \supseteq \cdots \supseteq C^\infty(I)$$

Example: $I = \mathbb{R}$
$f(x) = x^2$
exp

Proposition: $f: [a,b] \to \mathbb{R}$ differentiable, $x_0 \in [a,b]$, $f'(x_0) = 0$, and f' differentiable at x_0. Then: (a) $f''(x_0) > 0 \Rightarrow f$ has a local minimum at x_0

(b) $f''(x_0) < 0 \Rightarrow f$ has a local maximum at x_0

Proof: (a) Assume $0 < f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \Delta_{f', x_0}(x)$ continuous at x_0

$\Rightarrow$ There is a neighbourhood of x_0, called $U \subseteq [a,b]$, with $\Delta_{f', x_0}(x) > 0$

$\Rightarrow \left\{ 0 < \frac{f'(x)}{x - x_0} \right.$ for $x \in U \setminus \{x_0\}$

- $x < x_0 \Rightarrow f'(x) < 0 \Rightarrow f$ decreasing
- $x > x_0 \Rightarrow f'(x) > 0 \Rightarrow f$ increasing