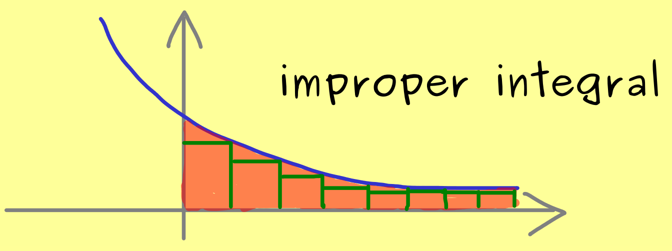
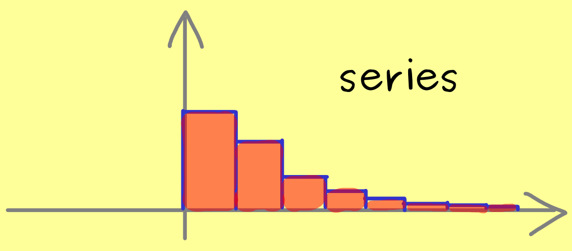
The Bright Side of Mathematics

Real Analysis – Part 62

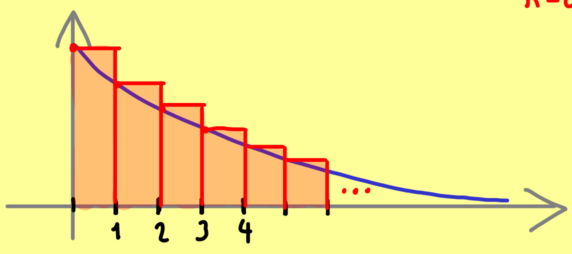


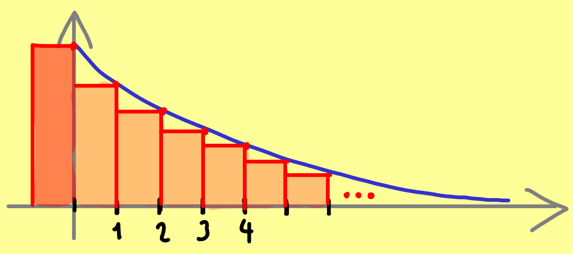


Theorem: Let $f : [0,\infty) \longrightarrow [0,\infty)$ be monotonically decreasing.

Then: $\sum_{k=0}^{\infty} f(k)$ convergent $\Longleftrightarrow$ $\int_0^{\infty} f(x)\,dx$ convergent

In this case: $0 \leq \sum_{k=0}^{\infty} f(k) - \int_0^{\infty} f(x)\,dx \leq f(0)$





Proof: $f(k) = \int_{k-1}^{k} f(k)\,dx \leq \int_{k-1}^{k} f(x)\,dx \leq \int_{k-1}^{k} f(k-1)\,dx = f(k-1)$

$$\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} \int_{k-1}^{k} f(x)\,dx \leq \sum_{k=1}^{n} f(k-1)$$

$$\Longrightarrow \sum_{k=1}^{n} f(k) \leq \int_0^{n} f(x)\,dx \leq \sum_{k=0}^{n-1} f(k)$$ ($n \to \infty$ shows first part)

If the limits exist: $\sum_{k=1}^{\infty} f(k) \leq \int_0^{\infty} f(x)\,dx \leq \sum_{k=0}^{\infty} f(k)$ $\square$

Example:

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \begin{cases} \text{convergent for } \alpha > 1 \\ \text{divergent for } 0 < \alpha \leq 1 \end{cases}$$

Proof:

$$\int_1^b \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{1-\alpha} x^{-\alpha+1} \Big|_1^b, & \alpha \neq 1 \\ \log(x) \Big|_1^b, & \alpha = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{1-\alpha} b^{-(\alpha-1)} - \frac{1}{1-\alpha}, & \alpha > 1 \\ \frac{1}{1-\alpha} b^{1-\alpha} - \frac{1}{1-\alpha}, & \alpha < 1 \\ \log(b), & \alpha = 1 \end{cases} \xrightarrow{b \to \infty} \begin{cases} \frac{1}{\alpha-1}, & \alpha > 1 \\ \infty, & \alpha < 1 \\ \infty, & \alpha = 1 \end{cases}$$