The Bright Side of Mathematics

Spectral Theory - Part 1

Functional Analysis | Hilbert Spaces | Measure Theory | Spectral Theory | Abstract Linear Algebra

From Abstract Linear Algebra - Part 47: Let $A \in \mathbb{C}^{n \times n}$ be self-adjoint $(A^* = A)$.

Then there is a unitary $U \in \mathbb{C}^{n \times n}$ with $U^* A U = \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}$ eigenvalues of A (repeated by multiplicities)

Definition: Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra on X.

Then a map $\mu : \mathcal{A} \longrightarrow \mathbb{C}$ is called a complex measure if

(1) $\mu(\emptyset) = 0$

(2) For every countable collection $A_i \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$,

we have $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ and sum is absolutely convergent.

Note: A complex measure is finite, meaning $|\mu(X)| < \infty$.

Example: $X = [0,1]$, $\lambda : \mathcal{B}([0,1]) \longrightarrow [0,\infty]$, $\delta_{\{0\}} : \mathcal{B}([0,1]) \longrightarrow [0,\infty]$

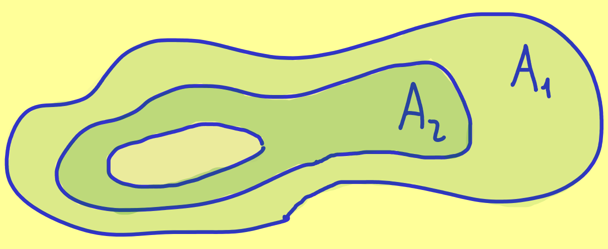
Borel σ-algebra   Lebesgue measure   Dirac measure

Then $\mu = \lambda + i\,\delta_{\{0\}}$ is a complex measure with $\mu([0,1]) = 1 + i$.

Proposition: Any complex measure is $\emptyset$-continuous, which means:

$$A_1 \supseteq A_2 \supseteq \cdots \text{ with } \bigcap_{j=1}^{\infty} A_j = \emptyset \implies \mu(A_n) \xrightarrow{n \to \infty} 0$$

Proof:



$C_k := A_k \setminus A_{k+1} \rightsquigarrow \bigcup_{k=1}^{\infty} C_k = A_1$

$$\sum_{k=1}^{n} \underbrace{\mu(C_k)}_{\mu(A_k) - \mu(A_{k+1})} \overset{\text{telescoping}}{=} \mu(A_1) - \mu(A_n)$$

We get: $$\mu(A_1) = \mu\left(\bigcup_{k=1}^{\infty} C_k\right) \underset{\sigma\text{-additivity}}{=} \sum_{k=1}^{\infty} \mu(C_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(C_k)$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n) \implies \lim_{n\to\infty} \mu(A_n) = 0 \qquad \square$$