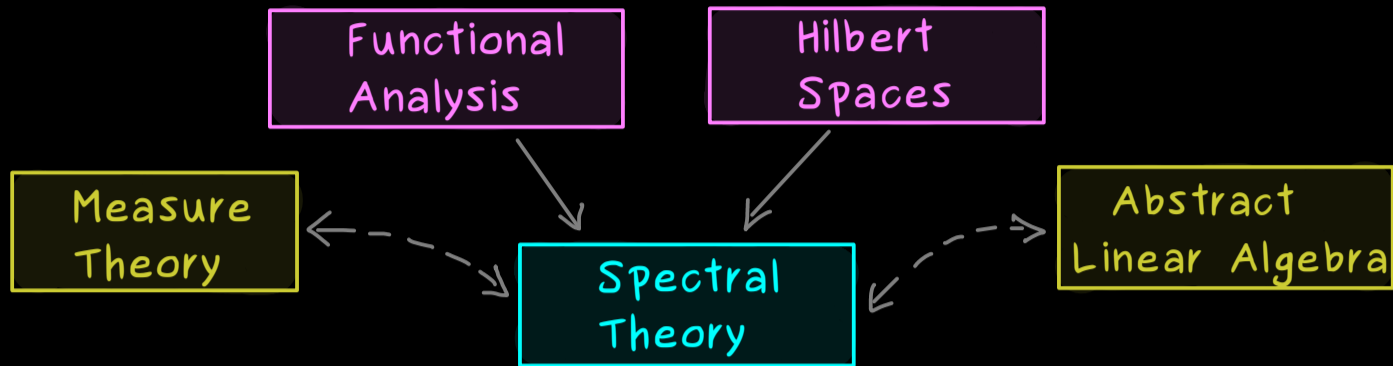


Spectral Theory - Part 1



From Abstract Linear Algebra - Part 47: Let $A \in \mathbb{C}^{n \times n}$ be self-adjoint ($A^* = A$).

Then there is a unitary $U \in \mathbb{C}^{n \times n}$ with $U^* A U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ eigenvalues of A (repeated by multiplicities)

Definition: Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ -algebra on X .

Then a map $\mu: \mathcal{A} \rightarrow \mathbb{C}$ is called a complex measure if

(1) $\mu(\emptyset) = 0$

(2) For every countable collection $A_i \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$,

we have $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ and sum is absolutely convergent.

Note: A complex measure is finite, meaning $|\mu(X)| < \infty$.

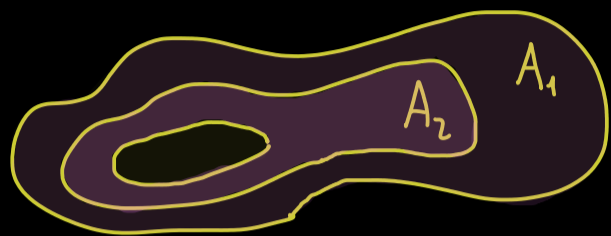
Example: $X = [0, 1]$, $\lambda: \mathcal{B}([0, 1]) \rightarrow [0, \infty]$, $\delta_{\{0\}}: \mathcal{B}([0, 1]) \rightarrow [0, \infty]$
↑ ↑ ↑
Borel σ -algebra Lebesgue measure Dirac measure

Then $\mu = \lambda + i \delta_{\{0\}}$ is a complex measure with $\mu([0, 1]) = 1 + i$.

Proposition: Any complex measure is ϕ -continuous, which means:

$$A_1 \supseteq A_2 \supseteq \dots \text{ with } \bigcap_{j=1}^{\infty} A_j = \emptyset \implies \mu(A_n) \xrightarrow{n \rightarrow \infty} 0$$

Proof:



$$C_k := A_k \setminus A_{k+1} \rightsquigarrow \bigcup_{k=1}^{\infty} C_k = A_1$$

$$\sum_{k=1}^n \underbrace{\mu(C_k)}_{\mu(A_k) - \mu(A_{k+1})} \stackrel{\text{telescoping}}{=} \mu(A_1) - \mu(A_n)$$

$$\begin{aligned} \text{We get: } \mu(A_1) &= \mu\left(\bigcup_{k=1}^{\infty} C_k\right) \stackrel{\uparrow}{=} \sum_{k=1}^{\infty} \mu(C_k) \stackrel{\uparrow}{=} \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(C_k) \\ &= \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_n) \implies \lim_{n \rightarrow \infty} \mu(A_n) = 0 \quad \square \end{aligned}$$