The Bright Side of Mathematics

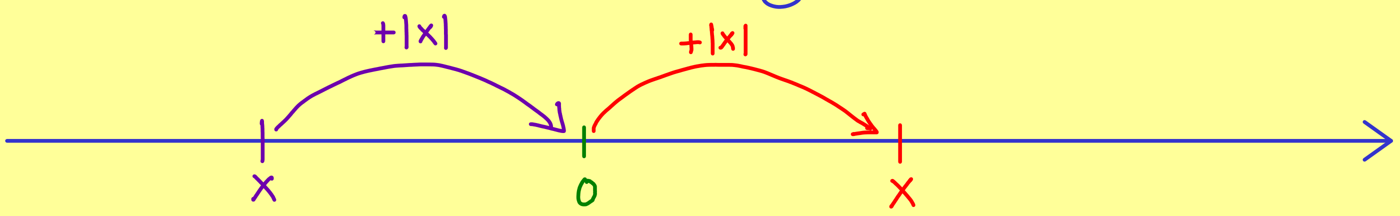
Start Learning Reals – Part 1

Real numbers $\mathbb{R}$

Starting point: $\mathbb{Q}$ is the set of fractions $\rightsquigarrow$ field and Archimedean order $\leq$

$$x > 0 \quad , \quad x < 0$$

Absolute value: For $x \in \mathbb{Q}$ define: $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



How far away is x from 0 ? $\rightsquigarrow |x|$

Problem: There is no $x \in \mathbb{Q}$ with $x^2 = 2$

$$x_1 = \frac{14}{10} = \frac{7}{5} \quad \rightsquigarrow \quad x_1^2 = \frac{49}{25} \approx 2$$

$$x_2 = \frac{141}{100} \quad \rightsquigarrow \quad x_2^2 = \frac{19\,881}{10\,000} \approx 2$$

$$x_3 = \frac{1414}{1000} \quad \rightsquigarrow \quad x_3^2 = \frac{499\,849}{250\,000} \approx 2$$

$$x_4 = \frac{14142}{10000} \quad \rightsquigarrow \quad x_4^2 = \frac{49\,999\,041}{25\,000\,000} \approx 2$$

$$x_5 = \frac{141\,421}{100\,000} \quad \rightsquigarrow \quad x_5^2 = \frac{19\,999\,899\,241}{10\,000\,000\,000} \approx 2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x = \,? \quad \rightsquigarrow \quad x^2 = 2$$

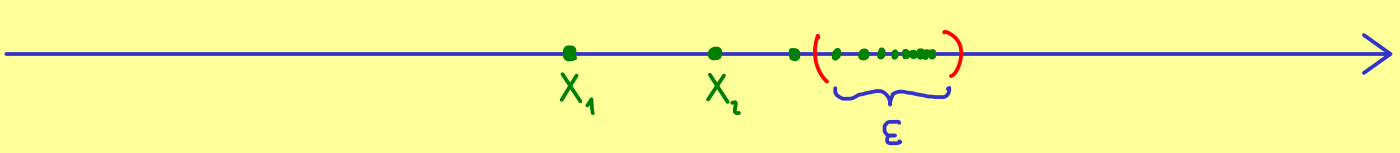
distance: $|x_5 - x_3|$

We consider a sequence $(x_n)_{n \in \mathbb{N}}$ (infinite list; formally: a map $\mathbb{N} \to \mathbb{Q}$, $n \mapsto x_n$)

with the property:

$$\forall \varepsilon \in \mathbb{Q} \quad \exists N \in \mathbb{N} \quad \forall n, m \in \mathbb{N} : \left(\varepsilon > 0 \ \wedge \ n, m \geq N \implies |x_n - x_m| < \varepsilon \right)$$

In short: $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq N : \quad |x_n - x_m| < \varepsilon \qquad (*)$



Cauchy sequence: sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{Q}$ and property $(*)$