**The Bright Side of Mathematics**

The following pages cover the whole Start Learning Reals course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

The Bright Side of Mathematics

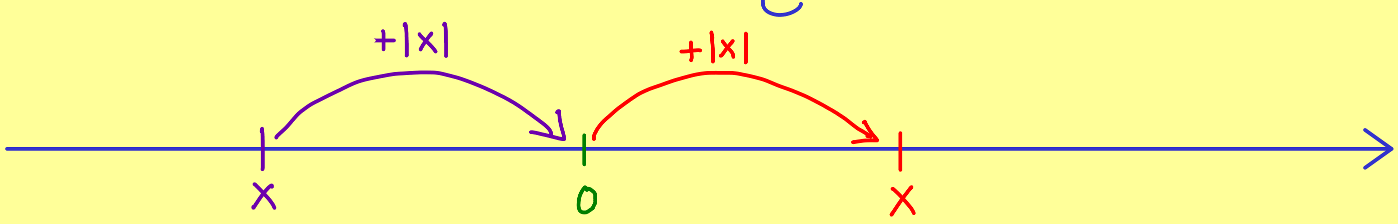
# Start Learning Reals – Part 1

⤷ Real numbers $\mathbb{R}$

**Starting point:** $\mathbb{Q}$ is the set of fractions $\rightsquigarrow$ field and Archimedean order $\leq$

$$x > 0 \quad , \quad x < 0$$

**Absolute value:** For $x \in \mathbb{Q}$ define: $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$




How far away is $x$ from $0$ ? $\rightsquigarrow |x|$

**Problem:** There is no $x \in \mathbb{Q}$ with $x^2 = 2$

$$x_1 = \frac{14}{10} = \frac{7}{5} \rightsquigarrow x_1^2 = \frac{49}{25} \approx 2$$

$$x_2 = \frac{141}{100} \rightsquigarrow x_2^2 = \frac{19\,881}{10\,000} \approx 2$$

$$x_3 = \frac{1414}{1000} \rightsquigarrow x_3^2 = \frac{499\,849}{250\,000} \approx 2$$

distance:

$|x_5 - x_3|$

$$x_4 = \frac{14142}{10000} \rightsquigarrow x_4^2 = \frac{49\,999\,041}{25\,000\,000} \approx 2$$

$$x_5 = \frac{141\,421}{100\,000} \rightsquigarrow x_5^2 = \frac{19\,999\,899\,241}{10\,000\,000\,000} \approx 2$$

$$\vdots \qquad \vdots \qquad \vdots$$

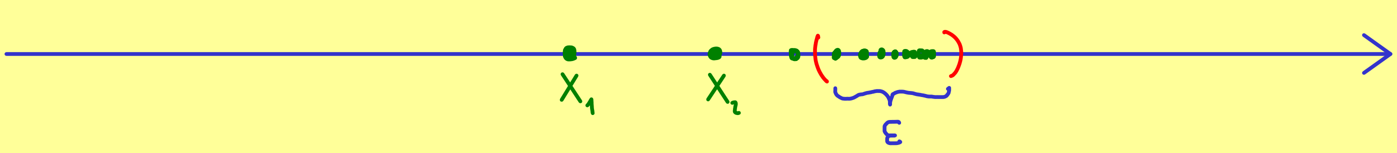
$$x = \,? \rightsquigarrow x^2 = 2$$

We consider a sequence $(x_n)_{n \in \mathbb{N}}$ (infinite list; formally: a map $\mathbb{N} \rightarrow \mathbb{Q}$ , $n \mapsto x_n$ )

with the property:

$$\forall \varepsilon \in \mathbb{Q} \quad \exists N \in \mathbb{N} \quad \forall n,m \in \mathbb{N} : \left( \varepsilon > 0 \ \wedge \ n,m \geq N \implies |x_n - x_m| < \varepsilon \right)$$

In short: $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n,m \geq N : \quad |x_n - x_m| < \varepsilon \qquad (*)$




**Cauchy sequence:** sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{Q}$ and property $(*)$

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Mathematics

# Start Learning Reals – Part 2

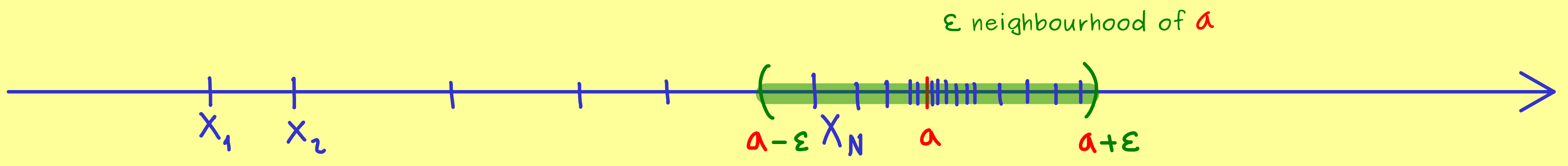
**Absolute value in $\mathbb{Q}$:** $|x \cdot y| = |x| \cdot |y|$ (multiplicative)

$|x + y| \leq |x| + |y|$ (triangle inequality)

**Cauchy sequence:** $(x_n)_{n \in \mathbb{N}}$ with $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n, m \geq N : \; |x_n - x_m| < \varepsilon$

**Convergent sequence:** $(x_n)_{n \in \mathbb{N}}$ with $\exists a \in \mathbb{Q} \;\; \forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N : \; |x_n - a| < \varepsilon$

$a$ is called the limit of $(x_n)_{n \in \mathbb{N}}$




**Example:** $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is a convergent sequence with limit $a = 0$.

**Important fact:** Cauchy sequence $\Leftarrow$ Convergent sequence

not correct $\mathbb{Q}$ but in $\mathbb{R}$

**Proof for $\Leftarrow$:** $|x_n - x_m| = |x_n - a + a - x_m| \overset{\text{triangle inequality}}{\leq} |x_n - a| + |a - x_m|$

Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence with limit $a$.

Let $\varepsilon > 0$. Set $\varepsilon' := \frac{\varepsilon}{2} > 0$.

Since $(x_n)_{n \in \mathbb{N}}$ is convergent, there is $N \in \mathbb{N}$ such that:

$$\forall n \geq N : \; |x_n - a| < \varepsilon'$$

Therefore for all $n, m \geq N$:

$$|x_n - x_m| \leq \underbrace{|x_n - a|}_{< \varepsilon'} + \underbrace{|a - x_m|}_{< \varepsilon'} < 2 \cdot \varepsilon' = \varepsilon \quad \Rightarrow \quad (x_n)_{n \in \mathbb{N}} \text{ Cauchy sequence}$$

**Axiomatic solution:** A non-empty set $\mathbb{R}$ together with operations $+, \cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R} \setminus \{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence.

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The complete, whole, full number line $\mathbb{R}$

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# Start Learning Reals – Part 3

complete number line $\mathbb{R}$

**Axioms of the reals:** A non-empty set $\mathbb{R}$ together with operations $+\,,\,\cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x\cdot(y+z) = x\cdot y + x\cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence. $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

**Important facts:** There is a set with all these properties (Existence) (Construction) see next video

and it is uniquely determined by these properties. (Uniqueness) (Identification/ Isomorphism)

**Show:** For all $x \in \mathbb{R}$, we have: $0 \cdot x = 0$ (*) (by only using the axioms).

**Proof:**

$$0 \overset{(A)}{\underset{\text{inverse}}{=}} (0\cdot x) + (-0\cdot x) \overset{(A)}{\underset{\text{neutral}}{=}} ((0+0)\cdot x) + (-0\cdot x)$$

$$\overset{(D)}{=} (0\cdot x + 0\cdot x) + (-0\cdot x)$$

$$\overset{(A)}{\underset{\text{associativity}}{=}} 0\cdot x + (0\cdot x + (-0\cdot x)) \overset{(A)}{\underset{\text{inverse}}{=}} 0\cdot x + 0 \overset{(A)}{\underset{\text{neutral}}{=}} 0\cdot x$$

**Show:** For all $x \in \mathbb{R}$, we have: $(-1)\cdot x = -x$ (by only using the axioms).

**Proof:**

$$-x \overset{(A)}{\underset{\text{neutral}}{=}} 0 + (-x) \overset{(*)}{=} 0\cdot x + (-x) \overset{(A)}{\underset{\text{inverse}}{=}} ((-1)+1)\cdot x + (-x)$$

$$\overset{(D)}{=} (-1)\cdot x + 1\cdot x + (-x) \overset{(A),(M)}{=} (-1)\cdot x + 0 \overset{(A)}{\underset{\text{neutral}}{=}} (-1)\cdot x$$

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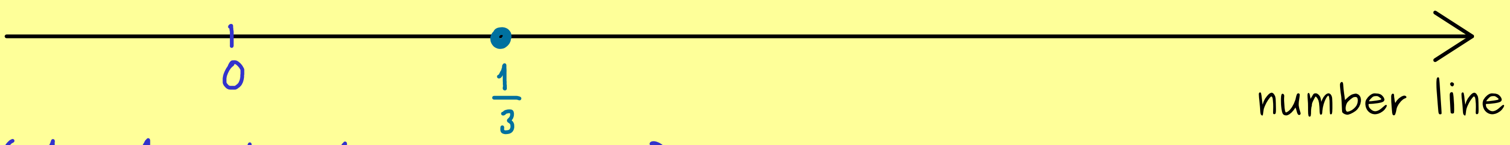
# Start Learning Reals – Part 4

Construction: $\mathbb{Q} \rightsquigarrow \mathbb{R}$ (Make every Cauchy sequence convergent)




Sequence: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\right)$ $\rightsquigarrow$ Cauchy sequence and convergent with limit $\frac{1}{3}$

Sequence: $\left(\underbrace{0.3}_{\frac{3}{10}}, \underbrace{0.33}_{\frac{33}{100}}, \underbrace{0.333}_{\frac{333}{1000}}, \dots\right)$ $\rightsquigarrow$ Cauchy sequence and convergent with limit $\frac{1}{3}$

$$\mathcal{C} := \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \forall n \in \mathbb{N}: x_n \in \mathbb{Q} \text{ and } (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence} \right\}$$

For two elements $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, define:

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \;:\Longleftrightarrow\; (a_n - b_n)_{n\in\mathbb{N}} \text{ convergent with limit } 0$$

$\Rightarrow$ $\sim$ is an equivalence relation on $\mathcal{C}$ (reflexive, symmetric, transitive)

$\Rightarrow$ equivalence class $\left[(x_n)_{n\in\mathbb{N}}\right]_\sim := \left\{ (a_n)_{n\in\mathbb{N}} \;\middle|\; (a_n)_{n\in\mathbb{N}} \sim (x_n)_{n\in\mathbb{N}} \right\}$

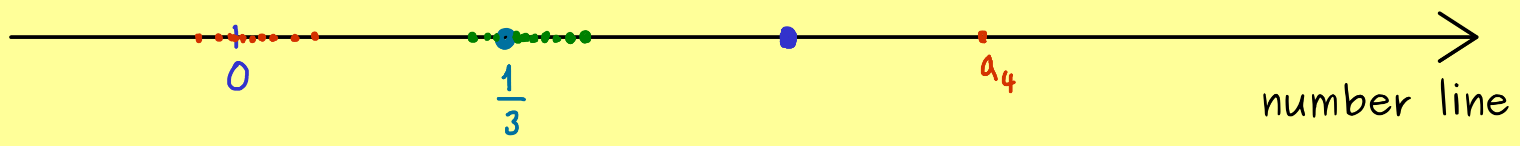
**Definition:**

$$\mathbb{R} := \mathcal{C}/_\sim := \left\{ \left[(x_n)_{n\in\mathbb{N}}\right]_\sim \;\middle|\; (x_n)_{n\in\mathbb{N}} \in \mathcal{C} \right\}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim + \left[(b_n)_{n\in\mathbb{N}}\right]_\sim := \left[(a_n + b_n)_{n\in\mathbb{N}}\right]_\sim \quad \text{(well-defined)}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim \cdot \left[(b_n)_{n\in\mathbb{N}}\right]_\sim := \left[(a_n \cdot b_n)_{n\in\mathbb{N}}\right]_\sim \quad \text{(well-defined)}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim < \left[(b_n)_{n\in\mathbb{N}}\right]_\sim \;:\Longleftrightarrow\; \exists \delta > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N: \; \delta < b_n - a_n$$




**Properties:**

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