The Bright Side of Mathematics

Start Learning Reals – Part 2

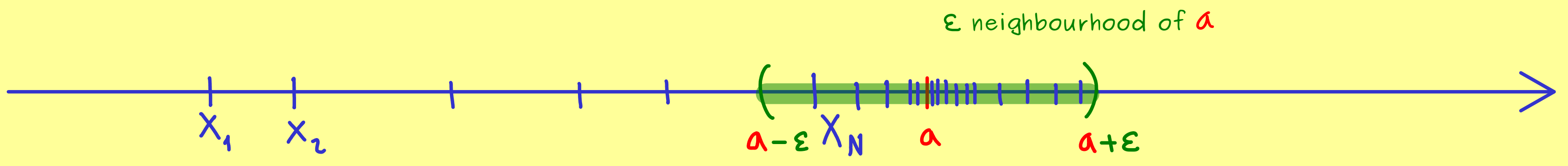
Absolute value in $\mathbb{Q}$: $|x \cdot y| = |x| \cdot |y|$ (multiplicative)

$|x + y| \leq |x| + |y|$ (triangle inequality)

Cauchy sequence: $(x_n)_{n \in \mathbb{N}}$ with $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n,m \geq N : |x_n - x_m| < \varepsilon$

Convergent sequence: $(x_n)_{n \in \mathbb{N}}$ with $\exists a \in \mathbb{Q} \; \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : |x_n - a| < \varepsilon$

a is called the limit of $(x_n)_{n \in \mathbb{N}}$



Example: $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is a convergent sequence with limit $a = 0$.

Important fact: Cauchy sequence $\Leftarrow$ Convergent sequence

$\Rightarrow$ not correct $\mathbb{Q}$ but in $\mathbb{R}$

Proof for $\Leftarrow$: $|x_n - x_m| = |x_n - a + a - x_m| \overset{\text{triangle inequality}}{\leq} |x_n - a| + |a - x_m|$

Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence with limit a.

Let $\varepsilon > 0$. Set $\varepsilon' := \frac{\varepsilon}{2} > 0$.

Since $(x_n)_{n \in \mathbb{N}}$ is convergent, there is $N \in \mathbb{N}$ such that:

$$\forall n \geq N : |x_n - a| < \varepsilon'$$

Therefore for all $n,m \geq N$:

$$|x_n - x_m| \leq \underbrace{|x_n - a|}_{< \varepsilon'} + \underbrace{|a - x_m|}_{< \varepsilon'} < 2 \cdot \varepsilon' = \varepsilon \quad \Rightarrow (x_n)_{n \in \mathbb{N}} \text{ Cauchy sequence}$$

Axiomatic solution: A non-empty set $\mathbb{R}$ together with operations $+, \cdot$ and ordering $\leq$ is called the real numbers if it satisfies:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R} \setminus \{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence.

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The complete, whole, full number line $\mathbb{R}$