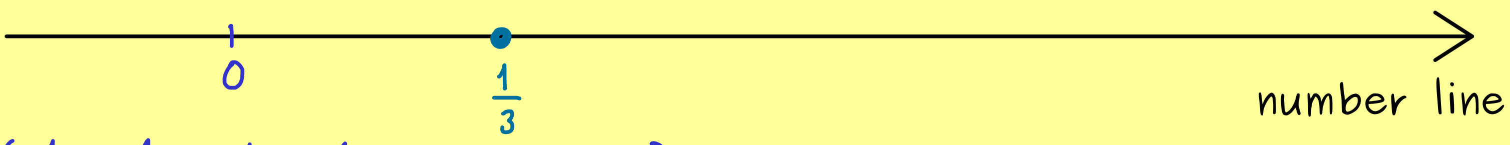
BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Start Learning Reals – Part 4

Construction: $\mathbb{Q} \rightsquigarrow \mathbb{R}$ (Make every Cauchy sequence convergent)



Sequence: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\right)$ $\rightsquigarrow$ Cauchy sequence and convergent with limit $\frac{1}{3}$

Sequence: $\left(\underbrace{0.3}_{\frac{3}{10}}, \underbrace{0.33}_{\frac{33}{100}}, \underbrace{0.333}_{\frac{333}{1000}}, \dots\right)$ $\rightsquigarrow$ Cauchy sequence and convergent with limit $\frac{1}{3}$

$$\mathcal{C} := \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \forall n \in \mathbb{N}: x_n \in \mathbb{Q} \text{ and } (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence} \right\}$$

For two elements $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, define:

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \quad :\Longleftrightarrow \quad (a_n - b_n)_{n\in\mathbb{N}} \text{ convergent with limit } 0$$

$\Rightarrow$ $\sim$ is an equivalence relation on $\mathcal{C}$ (reflexive, symmetric, transitive)

$\Rightarrow$ equivalence class $\left[(x_n)_{n\in\mathbb{N}}\right]_\sim := \left\{ (a_n)_{n\in\mathbb{N}} \;\middle|\; (a_n)_{n\in\mathbb{N}} \sim (x_n)_{n\in\mathbb{N}} \right\}$

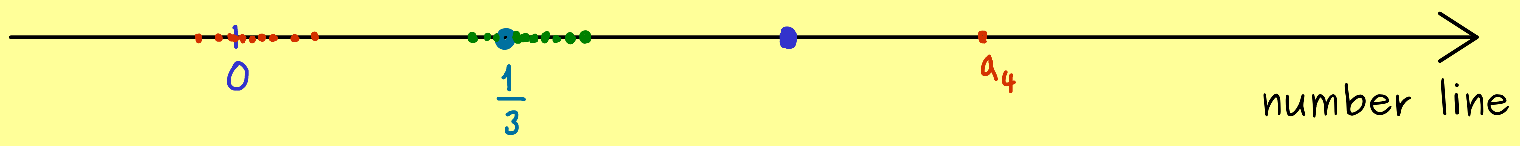
Definition:

$$\mathbb{R} := \mathcal{C}/_\sim := \left\{ \left[(x_n)_{n\in\mathbb{N}}\right]_\sim \;\middle|\; (x_n)_{n\in\mathbb{N}} \in \mathcal{C} \right\}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim + \left[(b_n)_{n\in\mathbb{N}}\right]_\sim := \left[(a_n + b_n)_{n\in\mathbb{N}}\right]_\sim \quad \text{(well-defined)}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim \cdot \left[(b_n)_{n\in\mathbb{N}}\right]_\sim := \left[(a_n \cdot b_n)_{n\in\mathbb{N}}\right]_\sim \quad \text{(well-defined)}$$

$$\left[(a_n)_{n\in\mathbb{N}}\right]_\sim < \left[(b_n)_{n\in\mathbb{N}}\right]_\sim \quad :\Longleftrightarrow \quad \exists \delta > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N: \; \delta < b_n - a_n$$



Properties:

(A) $(\mathbb{R}, +, 0)$ is an abelian group

(M) $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ is an abelian group $(1 \neq 0)$

(D) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$

(O) $\leq$ is a total order, compatible with $+$ and $\cdot$, Archimedean property

(C) Every Cauchy sequence is a convergent sequence. $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$