The Bright Side of Mathematics

The following pages cover the whole Unbounded Operators course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Unbounded Operators – Part 1

Linear Algebra   Real Analysis

Functional Analysis

Motivation:
- partial differential equations
- quantum mechanics: one needs operators X, P with
$$XP - PX = i \cdot I$$

Definition: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces (same field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$)

and $\mathcal{D} \subseteq X$ subspace.

A linear map $T : \mathcal{D} \longrightarrow Y$ is called an operator.

Other notations:
- $T : X \supseteq \mathcal{D} \longrightarrow Y$
- $T : X \longrightarrow Y$ with domain $\mathcal{D}$
- $(T, \mathcal{D})$ or T with $\mathcal{D}(T) = \mathcal{D}$

Moreover: T is called densely defined if $\overline{\mathcal{D}}^{\|\cdot\|_X} = X$.

$$\mathrm{Ran}(T) := \{ Tx \mid x \in \mathcal{D} \} \subseteq Y \quad \text{subspace}$$
$$\mathrm{Ker}(T) := \{ x \in \mathcal{D} \mid Tx = 0 \} \subseteq X \quad \text{subspace}$$

T is called bounded if $\exists C > 0 \;\; \forall x \in \mathcal{D} : \|Tx\|_Y \leq C \cdot \|x\|_X$

T is called unbounded if $\forall C > 0 \;\; \exists x \in \mathcal{D} : \|Tx\|_Y > C \cdot \|x\|_X$

Recall: T is bounded $\Longleftrightarrow$ T is continuous at all points $x \in \mathcal{D}$

Therefore: T is unbounded $\Longleftrightarrow$ T is not continuous (at no point $x \in \mathcal{D}$)

Unbounded Operators – Part 2

Recall: operator $T: X \to Y$ with $\mathcal{D}(T) = \mathcal{D}$

means: $T: \mathcal{D} \to Y$ linear map

Fact: If $\mathrm{Ker}(T) = \{0\}$, then $T^{-1}: Y \to X$ with $\mathcal{D}(T^{-1}) = \mathrm{Ran}(T)$

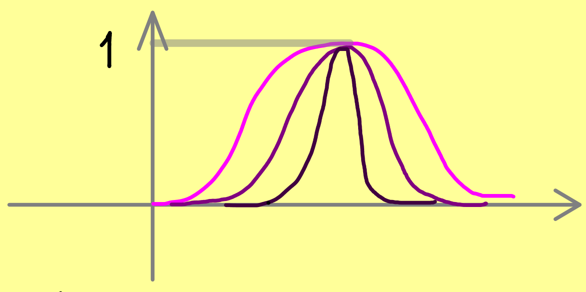
always defined as an operator

Examples: $X = Y = C([0,1])$ (with supremum norm $\|\cdot\|_\infty$)

(a) $T: X \to Y$ with $\mathcal{D}(T) = C^1([0,1])$

$Tx = x'$

unbounded operator



$$\|T\| = \sup_{\|x\|_\infty = 1} \|Tx\|_\infty = \sup_{\|x\|_\infty = 1} \|x'\|_\infty = \infty$$

(b) $S: X \to Y$ with $\mathcal{D}(S) = \{ x \in C^1([0,1]) \mid x(0) = 0 \}$

$Sx = x'$

notations: $S \subseteq T$

the operator T is an extension of S

the operator S is a restriction of T

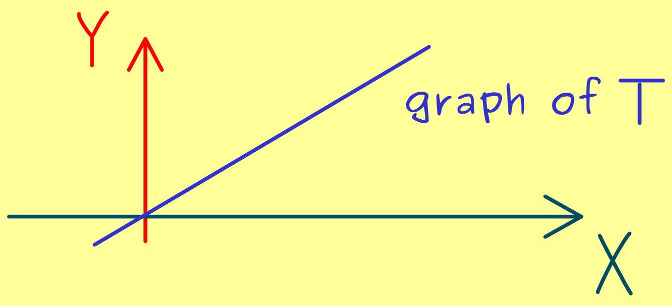
Note:

- $\mathrm{Ker}(T) \neq \{0\}$ not injective!
- $\mathrm{Ker}(S) = \{0\}$ injective! $\Rightarrow$ S^{-1} exists
- T is densely defined $\left(\overline{C^1([0,1])}^{\|\cdot\|_\infty} = C([0,1]) \right)$
- S is not densely defined

Unbounded Operators – Part 3

Recall: operator $T: X \supseteq D(T) \to Y$ (linear map between normed spaces)

$\leftsquigarrow\!\rightsquigarrow$ subset in $X \times Y$



graph of T : $G_T := \{ (x,y) \in X \times Y \mid x \in D(T) \,,\, Tx = y \}$

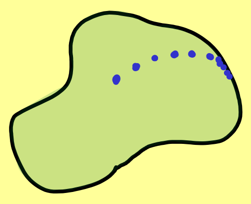
$X \times Y$ normed space with $\|(x,y)\|_{X \times Y} := \|x\|_X + \|y\|_Y$

Definition: An operator $T: X \supseteq D(T) \to Y$ is called a closed operator if the graph G_T is closed (in the normed space $X \times Y$).

Note: T closed $\Longleftrightarrow$ $\boxed{\begin{array}{l} \text{for each sequence } (x_n) \subseteq D(T) \text{ with} \\ x_n \xrightarrow{n \to \infty} x \in X \,,\; Tx_n \to y \in Y \,, \\ \text{we have: } x \in D(T) \text{ and } Tx = y \end{array}}$

Proof: G_T closed $\Longleftrightarrow$ for each sequence $(x_n, Tx_n) \subseteq G_T$ that is convergent in $X \times Y$ with limit $(x, y) \in X \times Y$,

we have: $\underbrace{(x, y) \in G_T}_{x \in D(T) \text{ and } Tx = y}$.



Remember: $T: X \to Y$ with $D(T) = X$ bounded $\Longrightarrow$ closed operator

Unbounded Operators – Part 4

Closed operator: $T : X \supseteq \mathcal{D}(T) \longrightarrow Y$ closed

$\Longleftrightarrow$ $G_T := \{(x,y) \in X \times Y \mid x \in \mathcal{D}(T),\ Tx = y\}$ closed

Closable operator: $T : X \supseteq \mathcal{D}(T) \longrightarrow Y$ closable

$:\Longleftrightarrow$ $\overline{G_T}$ is the graph of an operator $\overline{T}$ — closure of T

Proposition: $T : X \supseteq \mathcal{D}(T) \longrightarrow Y$ closable

$\Longleftrightarrow$ $\overline{G_T}$ is a graph (not possible $(0,0), (0,y) \in \overline{G_T}$ for $y \neq 0$)

$\Longleftrightarrow$ If $(0,y) \in \overline{G_T}$, then $y = 0$. $\boxed{G_T := \{(x,y) \in X \times Y \mid x \in \mathcal{D}(T),\ Tx = y\}}$

$\Longleftrightarrow$ For each $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have $y = 0$.

Define $\overline{T}$ for a closable operator $T : X \supseteq \mathcal{D}(T) \longrightarrow Y$:

$$\mathcal{D}(\overline{T}) := \{x \in X \mid \exists (x_n) \subseteq \mathcal{D}(T) : x_n \to x \text{ and } Tx_n \text{ convergent}\}$$

$$\overline{T}x := \lim_{n \to \infty} Tx_n$$ operator! (closure of T)

$\Longrightarrow$ $T \subseteq \overline{T}$

Unbounded Operators – Part 5

$T: X \supseteq \mathcal{D}(T) \longrightarrow Y$ closable $\Longleftrightarrow$ For each $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have: $y = 0$.

Example: $X = \ell^2(\mathbb{N}, \mathbb{C})$, $e_1, e_2, e_3, \ldots$ canonical unit vectors

$e_2 = (0,1,0,0,\ldots)$

$T: X \supseteq \mathcal{D}(T) \longrightarrow \mathbb{C}$, $\mathcal{D}(T) = \operatorname{span}\{e_j \mid j \in \mathbb{N}\}$

$$e_j \longmapsto j$$

$$\sum_j \lambda_j e_j \longmapsto \sum_j \lambda_j \cdot j$$

$$\|T\| = \sup_{\|x\|_X = 1} \|Tx\|_{\mathbb{C}} \geq \sup_{j \in \mathbb{N}} |Te_j| = \sup_{j \in \mathbb{N}} j = \infty$$

$\boxed{\text{unbounded operator!}}$ $\rightarrow$ not continuous at 0

Closable operator?

Choose $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \not\to 0$.

Choose $\varepsilon > 0$ and subsequence (x_{n_k}) such that: $|Tx_{n_k}| \geq \varepsilon$

Define: $z_k := \dfrac{x_{n_k}}{Tx_{n_k}} \xrightarrow{k \to \infty} 0$

Then: $Tz_k = \underbrace{1}_{=y}$ for all $k \in \mathbb{N}$

$\Longrightarrow$ T is not closable

For each $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have: $y = 0$.

Unbounded Operators – Part 6

Closed Graph Theorem: X, Y Banach spaces, $T: X \supseteq D(T) \rightarrow Y$ operator with $D(T)$ closed (e.g. $D(T) = X$).

Then: T closed $\Longleftrightarrow$ T continuous (bounded)

Proof: Assume: $D(T) = X$.

$(\Leftarrow)$ Choose $(x_n) \subseteq D(T)$ with $x_n \rightarrow x \in X$ and $Tx_n \rightarrow y \in Y$

$$\overset{T \text{ continuous}}{\Longrightarrow} \quad y = \lim_{n\rightarrow\infty} T(x_n) = T\left(\lim_{n\rightarrow\infty} x_n\right) = Tx$$

$$\Longrightarrow \quad x \in D(T) \text{ and } Tx = y \quad \Longrightarrow \quad T \text{ closed}$$

$(\Rightarrow)$ Assume T is closed $\Longrightarrow$ G_T is closed in $X \times Y$ $\Longrightarrow$ $(G_T, \|\cdot\|_{X\times Y})$ Banach space

Define operators: $P_X : G_T \longrightarrow X$, $(x, y) \mapsto x$ and $P_Y : G_T \longrightarrow Y$, $(x,y) \mapsto y$ linear + bounded

$\underbrace{(x,y) \mapsto x}_{\text{bijective!}}$

$\boxed{\text{Bounded Inverse Theorem}}$

Functional Analysis – Part 27

$$\Longrightarrow \quad P_X^{-1} : X \longrightarrow G_T \text{ is continuous (bounded operator)}$$
$$x \longmapsto (x, Tx)$$

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "P_X^{-1}"'] & & Y \\ & G_T \arrow[ur, "P_Y"'] & \end{tikzcd}$$

$T = P_Y P_X^{-1}$ composition of continuous maps

$\Longrightarrow$ T continuous (bounded)

Unbounded Operators – Part 7

X, Y Banach spaces, $T: X \supseteq D(T) \to Y$ operator

Closed Graph Theorem: $D(T) = X \Rightarrow \big(T \text{ closed} \Leftrightarrow T \text{ bounded} \big)$

Example: functional $T: X \to \mathbb{C}$ unbounded (see part 5)

extend: $D(T) = X$

$\Rightarrow$ T not closed

Proposition: X, Y Banach spaces, $T: X \supseteq D(T) \to Y$ operator.

Then: T closed $\Leftrightarrow$ $\big(D(T), \|\cdot\|_T \big)$ complete

graph norm

$$\|x\|_T := \|x\|_X + \|Tx\|_Y$$

Proof: $J: \big(D(T), \|\cdot\|_T \big) \longrightarrow \big(G_T, \|\cdot\|_{X \times Y} \big)$, $x \longmapsto (x, Tx)$ linear + bijective

$$\|Jx\|_{X\times Y} = \|(x, Tx)\|_{X\times Y} = \|x\|_X + \|Tx\|_Y = \|x\|_T$$

$\Rightarrow$ J is an isometric isomorphism

$\big(D(T), \|\cdot\|_T \big)$ complete $\Leftrightarrow$ $\big(G_T, \|\cdot\|_{X\times Y} \big)$ complete

$\Leftrightarrow$ $\big(G_T, \|\cdot\|_{X\times Y} \big)$ closed in $X \times Y$

$\Leftrightarrow$ T closed

Unbounded Operators – Part 8

For bounded operators: $T: X \to Y$ $\rightsquigarrow$ $T^*: Y \to X$ adjoint

Hilbert spaces

$$\langle y, Tx \rangle_Y = \langle T^* y, x \rangle_X$$

$T: X \to Y$ $\rightsquigarrow$ $T': Y' \to X'$ adjoint

Banach spaces

$$T'(y')(x) = y'(Tx)$$

for $y' \in Y'$, $x \in X$

Proposition: X, Y Banach spaces, $T: X \supseteq D(T) \to Y$ densely defined operator $\hookrightarrow \overline{D(T)} = X$

Then there is an operator $T': Y' \supseteq D(T') \to X'$ with

$$y'(Tx) = T'(y')(x) \quad \text{for } x \in D(T),\ y' \in D(T').$$

The domain $D(T')$ can be chosen maximally.

Proof: Set $D(T') := \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in D(T) \}$

and define: $T'(y') := x'$

Well-defined? Assume there are $x_1', x_2' \in X'$ with $y'(Tx) = x_1'(x)$, $y'(Tx) = x_2'(x)$ for all $x \in D(T)$

$$\Rightarrow x_1'(x) = x_2'(x) \quad \text{for all } x \in D(T)$$

$$\Rightarrow (x_1' - x_2')(x) = 0 \quad \text{for all } x \in D(T) \quad \overset{\text{dense}}{\underset{\text{continuity}}{\Longrightarrow}} \quad (x_1' - x_2')(x) = 0 \quad \text{for all } x \in X$$

$$\Rightarrow x_1' = x_2' \qquad \square$$

For Hilbert spaces: X, Y Hilbert spaces, $T: X \supseteq D(T) \to Y$ densely defined operator $\hookrightarrow \overline{D(T)} = X$

$$D(T^*) := \{ y \in Y \mid \text{there is } \tilde{x} \in X \text{ with } \langle y, Tx \rangle_Y = \langle \tilde{x}, x \rangle_X \text{ for all } x \in D(T) \}$$

$$T^*(y) := \tilde{x}$$

Unbounded Operators - Part 9

X, Y Banach spaces

$T: X \supseteq \mathcal{D}(T) \to Y$

densely defined operator

$\Longrightarrow \quad T': Y' \supseteq \mathcal{D}(T') \to X'$

(Banach space) adjoint operator

X, Y Hilbert spaces

$T: X \supseteq \mathcal{D}(T) \to Y$

densely defined operator

$\Longrightarrow \quad T^*: Y \supseteq \mathcal{D}(T^*) \to X$

(Hilbert space) adjoint operator

Connection between T' and T^*:

$$\begin{tikzcd} \boxed{Y} \arrow[r, bend left, "T^*"] \arrow[d, "C_Y"] & \boxed{X} \arrow[d, "C_X"] \\ \boxed{Y'} \arrow[r, bend left, "T'"] & \boxed{X'} \end{tikzcd}$$

Riesz representation theorem: $X' \cong X$ antilinear isometric isomorphism

(for Hilbert spaces)

$C_X : X \to X', \quad x \mapsto \langle x, \cdot \rangle_X = \langle x|$

$C_Y : Y \to Y', \quad y \mapsto \langle y, \cdot \rangle_Y = \langle y|$

where $T'(\langle y|)(x) = \langle y, Tx \rangle_Y$

for $x \in \mathcal{D}(T)$ $\quad = \langle T^* y, x \rangle_Y$

$$C_X^{-1} T' C_Y(y) = C_X^{-1} T'(\langle y|) \quad \text{for } y \in \mathcal{D}(T^*)$$

$$= C_X^{-1}(\langle T^* y|)$$

$$= T^* y$$

$$\Longrightarrow \quad T^* = C_X^{-1} T' C_Y$$

Proposition: X, Y Banach spaces, $T: X \supseteq \mathcal{D}(T) \to Y$ densely defined operator.

Then: $T \subseteq S \quad \Longrightarrow \quad T' \supseteq S'$

$\left(\mathcal{D}(T) \subseteq \mathcal{D}(S), \ S \text{ extension of } T, \ Sx = Tx \text{ for all } x \in \mathcal{D}(T) \right)$ $\quad \left(\mathcal{D}(T') \supseteq \mathcal{D}(S'), \ S' \text{ restriction of } T', \ S'y' = T'y' \text{ for all } y' \in \mathcal{D}(S') \right)$

And for Hilbert spaces: $T \subseteq S \quad \Longrightarrow \quad T^* \supseteq S^*$

Proof: $\mathcal{D}(S') := \left\{ y' \in Y' \ \middle| \ \text{there is } x' \in X' \text{ with } y'(Sx) = x'(x) \text{ for all } x \in \mathcal{D}(S) \right\}$

$$\subseteq \left\{ y' \in Y' \ \middle| \ \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in \mathcal{D}(T) \right\}$$

$$= \mathcal{D}(T') \qquad \square$$

Unbounded Operators - Part 10

$T: X \supseteq D(T) \to Y$ densely defined $\Longrightarrow$ adjoint exists:

(X, Y Banach spaces) $T': Y' \supseteq D(T') \to X'$

(X, Y Hilbert spaces) $T^*: Y \supseteq D(T^*) \to X$

Definition: Let $X = L^2(\mathbb{R}, \mathbb{C})$ ← square-integrable functions $\int_{\mathbb{R}} |f(x)|^2 \, dx < \infty$ with respect to the one-dimensional Lebesgue measure

Hilbert space with inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} \, g(x) \, dx$$

Let $\varphi: \mathbb{R} \to \mathbb{C}$ be a continuous function.

Then $M_\varphi: X \supseteq D(M_\varphi) \to X$ denotes the multiplication operator:

$f \mapsto M_\varphi f$ with $(M_\varphi f)(x) = \varphi(x) f(x)$ for $x \in \mathbb{R}$ almost everywhere

$$D(M_\varphi) := \{ f \in L^2(\mathbb{R}, \mathbb{C}) \mid \varphi \cdot f \in L^2(\mathbb{R}, \mathbb{C}) \}$$

dense in $L^2(\mathbb{R}, \mathbb{C})$

Adjoint of the multiplication operator: $(M_\varphi)^*: X \supseteq D((M_\varphi)^*) \to X$

$\{ g \in X \mid \text{there is } \tilde{f} \in X \text{ with } \langle g, M_\varphi f \rangle = \langle \tilde{f}, f \rangle \text{ for all } f \in D(M_\varphi) \}$ // with $(M_\varphi)^* g = \tilde{f}$

Is it a multiplication operator as well?

$$\langle g, M_\varphi f \rangle = \int_{\mathbb{R}} \overline{g(x)} \varphi(x) f(x) \, dx = \int_{\mathbb{R}} \overline{\overline{\varphi(x)} g(x)} \, f(x) \, dx = \langle M_{\overline{\varphi}} g, f \rangle$$

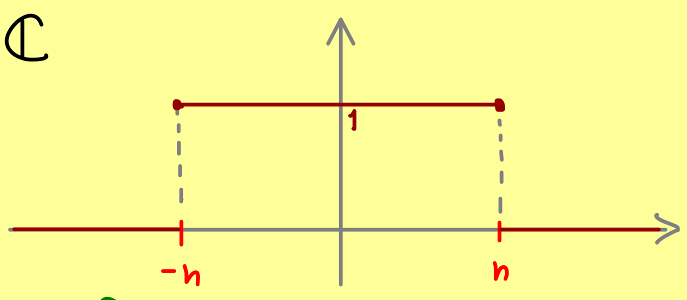
for all $f, g \in D(M_\varphi) = D(M_{\overline{\varphi}})$

First result: $M_{\overline{\varphi}} \subseteq (M_\varphi)^*$

To show: $g \in D((M_\varphi)^*) \Longrightarrow \overline{\varphi} \cdot g \in L^2(\mathbb{R}, \mathbb{C})$

Proof: Note: $g \in L^2$, h bounded $\Longrightarrow h \cdot g \in L^2$

Make $\overline{\varphi}$ bounded? Take $\chi_n: \mathbb{R} \to \mathbb{C}$



$\chi_n \overline{\varphi}$ is bounded

$(\chi_n \overline{\varphi})(x) \xrightarrow{n \to \infty} \overline{\varphi}(x)$ for $x \in \mathbb{R}$

For $f \in D(M_\varphi)$, $g \in D((M_\varphi)^*)$:

$$\langle \chi_n (M_\varphi)^* g, f \rangle = \int_{\mathbb{R}} \overline{\chi_n(x) (M_\varphi)^* g(x)} \, f(x) \, dx$$

$$= \langle (M_\varphi)^* g, \chi_n f \rangle = \langle g, M_\varphi (\chi_n f) \rangle$$

$$= \int_{\mathbb{R}} \overline{g(x)} \, \varphi(x) \chi_n(x) f(x) \, dx$$

$$= \int_{\mathbb{R}} \overline{\overline{\varphi(x)} \chi_n(x) g(x)} \, f(x) \, dx = \langle \chi_n \overline{\varphi} g, f \rangle$$

$$\overset{D(M_\varphi) \text{ dense}}{\Longrightarrow} \chi_n (M_\varphi)^* g = \chi_n \overline{\varphi} g \overset{\chi_n \xrightarrow{n \to \infty} 1}{\Longrightarrow} (M_\varphi)^* g = \overline{\varphi} g \in L^2$$

Final result: $(M_\varphi)^* = M_{\overline{\varphi}}$