Unbounded Operators – Part 6

Closed Graph Theorem: X, Y Banach spaces, $T: X \supseteq D(T) \longrightarrow Y$ operator with $D(T)$ closed (e.g. $D(T) = X$).

Then: T closed $\iff$ T continuous (bounded)

Proof: Assume: $D(T) = X$.

$(\Leftarrow)$ Choose $(x_n) \subseteq D(T)$ with $x_n \to x \in X$ and $Tx_n \to y \in Y$

$$\overset{T \text{ continuous}}{\Longrightarrow} \quad y = \lim_{n\to\infty} T(x_n) = T\left(\lim_{n\to\infty} x_n\right) = Tx$$

$$\Longrightarrow \quad x \in D(T) \text{ and } Tx = y \quad \Longrightarrow \quad T \text{ closed}$$

$(\Rightarrow)$ Assume T is closed $\Longrightarrow$ G_T is closed in $X \times Y$ $\Longrightarrow$ $(G_T, \|\cdot\|_{X \times Y})$ Banach space

Define operators:
$$P_X : G_T \longrightarrow X, \quad \underbrace{(x,y) \longmapsto x}_{\text{bijective!}} \qquad \text{and} \qquad P_Y : G_T \longrightarrow Y, \quad (x,y) \longmapsto y$$

linear + bounded

$\boxed{\text{Bounded Inverse Theorem}}$

Functional Analysis – Part 27

$$\Longrightarrow \quad P_X^{-1} : X \longrightarrow G_T \text{ is continuous (bounded operator)}$$
$$x \longmapsto (x, Tx)$$

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "P_X^{-1}"'] & & Y \\
& G_T \arrow[ur, "P_Y"'] &
\end{tikzcd}$$

$T = P_Y P_X^{-1}$ composition of continuous maps

$\Longrightarrow$ T continuous (bounded)