**The Bright Side of Mathematics**

The following pages cover the whole Unbounded Operators course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

# Unbounded Operators – Part 1

Linear Algebra | Real Analysis

Functional Analysis

**Motivation:**
- partial differential equations
- quantum mechanics: one needs operators $X, P$ with

$$XP - PX = i \cdot I$$

**Definition:** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces (same field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$)

and $\mathcal{D} \subseteq X$ subspace.

A linear map $T : \mathcal{D} \longrightarrow Y$ is called an operator.

Other notations:
- $T : X \supseteq \mathcal{D} \longrightarrow Y$
- $T : X \longrightarrow Y$ with domain $\mathcal{D}$
- $(T, \mathcal{D})$ or $T$ with $\mathcal{D}(T) = \mathcal{D}$

Moreover: $T$ is called densely defined if $\overline{\mathcal{D}}^{\|\cdot\|_X} = X$.

$\mathrm{Ran}(T) := \{ Tx \mid x \in \mathcal{D} \} \subseteq Y$ subspace

$\mathrm{Ker}(T) := \{ x \in \mathcal{D} \mid Tx = 0 \} \subseteq X$ subspace

$T$ is called bounded if $\exists C > 0 \ \forall x \in \mathcal{D} : \|Tx\|_Y \leq C \cdot \|x\|_X$

$T$ is called unbounded if $\forall C > 0 \ \exists x \in \mathcal{D} : \|Tx\|_Y > C \cdot \|x\|_X$

**Recall:** $T$ is bounded $\Longleftrightarrow$ $T$ is continuous at all points $x \in \mathcal{D}$

**Therefore:** $T$ is unbounded $\Longleftrightarrow$ $T$ is not continuous (at no point $x \in \mathcal{D}$ )

# Unbounded Operators – Part 2

**Recall:** operator $T: X \to Y$ with $\mathcal{D}(T) = \mathcal{D}$

means: $T: \mathcal{D} \to Y$ linear map

**Fact:** If $\mathrm{Ker}(T) = \{0\}$, then $T^{-1}: Y \to X$ with $\mathcal{D}(T^{-1}) = \mathrm{Ran}(T)$

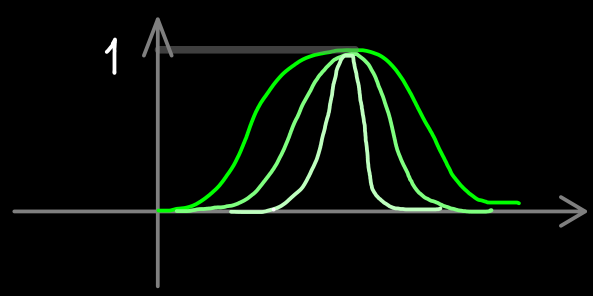
↳ always defined as an operator

**Examples:** $X = Y = C([0,1])$ (with supremum norm $\|\cdot\|_\infty$)

(a) $T: X \to Y$ with $\mathcal{D}(T) = C^1([0,1])$

$Tx = x'$

unbounded operator




$$\|T\| = \sup_{\|x\|_\infty = 1} \|Tx\|_\infty = \sup_{\|x\|_\infty = 1} \|x'\|_\infty = \infty$$

(b) $S: X \to Y$ with $\mathcal{D}(S) = \{ x \in C^1([0,1]) \mid x(0) = 0 \}$

$Sx = x'$

notations: $S \subseteq T$

the operator $T$ is an extension of $S$

the operator $S$ is a restriction of $T$

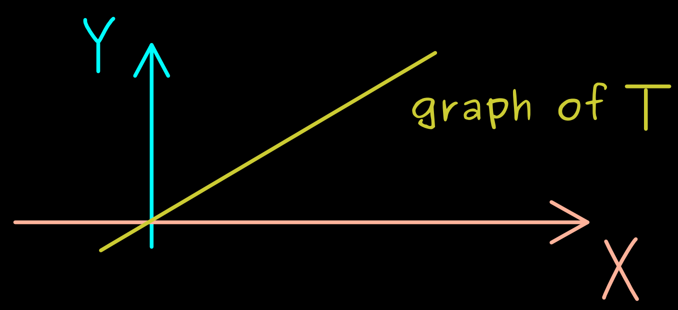
**Note:**

- $\mathrm{Ker}(T) \neq \{0\}$ not injective!
- $\mathrm{Ker}(S) = \{0\}$ injective! $\Rightarrow$ $S^{-1}$ exists
- $T$ is densely defined $\left( \overline{C^1([0,1])}^{\|\cdot\|_\infty} = C([0,1]) \right)$
- $S$ is not densely defined

# Unbounded Operators — Part 3

**Recall:** operator $T: X \supseteq D(T) \to Y$ (linear map between normed spaces)

$\leftrightsquigarrow$ subset in $X \times Y$



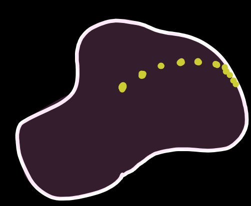

graph of $T$: $G_T := \{ (x,y) \in X \times Y \mid x \in D(T),\ Tx = y \}$

$X \times Y$ normed space with $\|(x,y)\|_{X \times Y} := \|x\|_X + \|y\|_Y$

**Definition:** An operator $T: X \supseteq D(T) \to Y$ is called a closed operator if the graph $G_T$ is closed (in the normed space $X \times Y$).

**Note:** $T$ closed $\Longleftrightarrow$ $\boxed{\begin{array}{l} \text{for each sequence } (x_n) \subseteq D(T) \text{ with} \\ \quad x_n \xrightarrow{n \to \infty} x \in X,\ Tx_n \to y \in Y, \\ \text{we have: } x \in D(T) \text{ and } Tx = y \end{array}}$

**Proof:** $G_T$ closed $\Longleftrightarrow$ for each sequence $(x_n, Tx_n) \subseteq G_T$ that is convergent in $X \times Y$ with limit $(x,y) \in X \times Y$, we have: $\underbrace{(x,y) \in G_T}_{x \in D(T) \text{ and } Tx = y}$.



**Remember:** $T: X \to Y$ with $D(T) = X$ bounded $\Rightarrow$ closed operator

# Unbounded Operators — Part 4

Closed operator: $T: X \supseteq \mathcal{D}(T) \longrightarrow Y$ closed

$$\Longleftrightarrow \quad G_T := \{(x,y) \in X \times Y \mid x \in \mathcal{D}(T),\ Tx = y\} \ \text{ closed}$$

Closable operator: $T: X \supseteq \mathcal{D}(T) \longrightarrow Y$ closable

$:\Longleftrightarrow \quad \overline{G_T}$ is the graph of an operator $\overline{T}$ closure of $T$

Proposition: $T: X \supseteq \mathcal{D}(T) \longrightarrow Y$ closable

$\Longleftrightarrow \quad \overline{G_T}$ is a graph (not possible $(0,0), (0,y) \in \overline{G_T}$ for $y \neq 0$)

$\Longleftrightarrow$ If $(0,y) \in \overline{G_T}$, then $y = 0$. $\boxed{G_T := \{(x,y) \in X \times Y \mid x \in \mathcal{D}(T),\ Tx = y\}}$

$\Longleftrightarrow$ For each $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have $y = 0$.

Define $\overline{T}$ for a closable operator $T: X \supseteq \mathcal{D}(T) \longrightarrow Y$:

$$\mathcal{D}(\overline{T}) := \{x \in X \mid \exists (x_n) \subseteq \mathcal{D}(T): x_n \to x \text{ and } Tx_n \text{ convergent}\}$$

$\overline{T}x := \lim\limits_{n \to \infty} Tx_n$ operator! (closure of $T$)

$$\Longrightarrow \quad T \subseteq \overline{T}$$

# Unbounded Operators — Part 5

$T: X \supseteq D(T) \longrightarrow Y$ closable $\Longleftrightarrow$ For each $(x_n) \subseteq D(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have: $y = 0$.

Example: $X = \ell^2(\mathbb{N}, \mathbb{C})$ , $e_1, e_2, e_3, \ldots$ canonical unit vectors

$e_2 = (0,1,0,0,\ldots)$

$$T: X \supseteq D(T) \longrightarrow \mathbb{C} \quad , \quad D(T) = \operatorname{span}\{e_j \mid j \in \mathbb{N}\}$$

$$e_j \longmapsto j$$

$$\sum_j \lambda_j e_j \longmapsto \sum_j \lambda_j \cdot j$$

$$\|T\| = \sup_{\|x\|_X = 1} \|Tx\|_{\mathbb{C}} \geq \sup_{j \in \mathbb{N}} |Te_j| = \sup_{j \in \mathbb{N}} j = \infty$$

$\boxed{\text{unbounded operator!}}$ $\Rightarrow$ not continuous at $0$

Closable operator?

Choose $(x_n) \subseteq D(T)$ with $x_n \to 0$ and $Tx_n \not\to 0$.

Choose $\varepsilon > 0$ and subsequence $(x_{n_k})$ such that: $|Tx_{n_k}| \geq \varepsilon$

Define: $z_k := \dfrac{x_{n_k}}{Tx_{n_k}} \xrightarrow{k \to \infty} 0$

Then: $Tz_k = 1$ for all $k \in \mathbb{N}$ ($1 = y$)

$\Longrightarrow$ $T$ is not closable

For each $(x_n) \subseteq D(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have: $y = 0$.

# Unbounded Operators – Part 6

**Closed Graph Theorem:** $X, Y$ Banach spaces, $T: X \supseteq D(T) \to Y$ operator with $D(T)$ closed $\big($e.g. $D(T) = X\big)$.

Then: $T$ closed $\Longleftrightarrow$ $T$ continuous (bounded)

**Proof:** Assume: $D(T) = X$.

$(\Leftarrow)$ Choose $(x_n) \subseteq D(T)$ with $x_n \to x \in X$ and $Tx_n \to y \in Y$

$$\overset{T \text{ continuous}}{\Longrightarrow} \quad y = \lim_{n \to \infty} T(x_n) = T\big(\lim_{n \to \infty} x_n\big) = Tx$$

$$\Longrightarrow \quad x \in D(T) \text{ and } Tx = y \quad \Longrightarrow \quad T \text{ closed}$$

$(\Rightarrow)$ Assume $T$ is closed $\Rightarrow$ $G_T$ is closed in $X \times Y$ $\Rightarrow$ $(G_T, \|\cdot\|_{X \times Y})$ Banach space

Define operators:

$$\underbrace{P_X : G_T \to X, \quad (x,y) \mapsto x}_{\text{bijective!}} \qquad \text{and} \qquad P_Y : G_T \to Y, \quad (x,y) \mapsto y$$

linear + bounded

$\boxed{\text{Bounded Inverse Theorem}}$

Functional Analysis – Part 27

$$\Longrightarrow \quad P_X^{-1} : X \to G_T, \quad x \mapsto (x, Tx) \quad \text{is continuous (bounded operator)}$$

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "P_X^{-1}"'] & & Y \\
& G_T \arrow[ur, "P_Y"'] &
\end{tikzcd}$$

$T = P_Y P_X^{-1}$ composition of continuous maps

$\Longrightarrow$ $T$ continuous (bounded)

# Unbounded Operators – Part 7

$X, Y$ Banach spaces, $T: X \supseteq \mathcal{D}(T) \to Y$ operator

Closed Graph Theorem: $\mathcal{D}(T) = X \Rightarrow \big( T$ closed $\Leftrightarrow T$ bounded $\big)$

Example: functional $T: X \to \mathbb{C}$ unbounded (see part 5)

extend: $\mathcal{D}(T) = X$

$\Rightarrow T$ not closed

Proposition: $X, Y$ Banach spaces, $T: X \supseteq \mathcal{D}(T) \to Y$ operator.

Then: $T$ closed $\Leftrightarrow \big(\mathcal{D}(T), \|\cdot\|_T\big)$ complete

graph norm

$$\|x\|_T := \|x\|_X + \|Tx\|_Y$$

Proof:

$$J: \big(\mathcal{D}(T), \|\cdot\|_T\big) \longrightarrow \big(G_T, \|\cdot\|_{X \times Y}\big)$$
$$x \longmapsto (x, Tx)$$

linear + bijective

$$\|Jx\|_{X\times Y} = \|(x, Tx)\|_{X\times Y} = \|x\|_X + \|Tx\|_Y = \|x\|_T$$

$\Rightarrow J$ is an isometric isomorphism

$\big(\mathcal{D}(T), \|\cdot\|_T\big)$ complete $\Leftrightarrow \big(G_T, \|\cdot\|_{X\times Y}\big)$ complete

$\Leftrightarrow \big(G_T, \|\cdot\|_{X\times Y}\big)$ closed in $X \times Y$

$\Leftrightarrow T$ closed

# Unbounded Operators – Part 8

**For bounded operators:** $T: X \to Y \rightsquigarrow T^*: Y \to X$ adjoint

Hilbert spaces

$$\langle y, Tx\rangle_Y = \langle T^* y, x\rangle_X$$

$T: X \to Y \rightsquigarrow T': Y' \to X'$ adjoint

Banach spaces

$$T'(y')(x) = y'(Tx)$$

for $y' \in Y'$, $x \in X$

**Proposition:** $X, Y$ Banach spaces, $T: X \supseteq D(T) \to Y$ densely defined operator $\hookrightarrow \overline{D(T)} = X$

Then there is an operator $T': Y' \supseteq D(T') \to X'$ with

$$y'(Tx) = T'(y')(x) \quad \text{for } x \in D(T),\ y' \in D(T').$$

The domain $D(T')$ can be chosen maximally.

**Proof:** Set $D(T') := \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in D(T) \}$

and define: $T'(y') := x'$

Well-defined? Assume there are $x_1', x_2' \in X'$ with
$$\left.\begin{aligned} y'(Tx) &= x_1'(x) \\ y'(Tx) &= x_2'(x) \end{aligned}\right\} \text{ for all } x \in D(T)$$

$$\Rightarrow x_1'(x) = x_2'(x) \quad \text{for all } x \in D(T)$$

$$\Rightarrow (x_1' - x_2')(x) = 0 \quad \text{for all } x \in D(T) \overset{\text{dense}}{\underset{\text{continuity}}{\Longrightarrow}} (x_1' - x_2')(x) = 0 \quad \text{for all } x \in X$$

$$\Rightarrow x_1' = x_2'$$

$\square$

For Hilbert spaces: $X, Y$ Hilbert spaces, $T: X \supseteq \mathcal{D}(T) \longrightarrow Y$ densely defined operator

$\hookrightarrow \overline{\mathcal{D}(T)} = X$

$$\mathcal{D}(T^*) := \left\{ y \in Y \;\middle|\; \text{there is } \tilde{x} \in X \text{ with } \langle y, Tx \rangle_Y = \langle \tilde{x}, x \rangle_X \text{ for all } x \in \mathcal{D}(T) \right\}$$

$$T^*(y) := \tilde{x}$$

# Unbounded Operators - Part 9

$X, Y$ Banach spaces

$T : X \supseteq D(T) \to Y$

densely defined operator

$\Longrightarrow \quad T' : Y' \supseteq D(T') \to X'$

(Banach space) adjoint operator

$X, Y$ Hilbert spaces

$T : X \supseteq D(T) \to Y$

densely defined operator

$\Longrightarrow \quad T^* : Y \supseteq D(T^*) \to X$

(Hilbert space) adjoint operator

## Connection between $T'$ and $T^*$:

$$\begin{tikzcd}
\boxed{Y} \arrow[r, bend left, "T^*"] \arrow[d, "C_Y"] & \boxed{X} \arrow[d, "C_X"] \\
\boxed{Y'} \arrow[r, bend left, "T'"] & \boxed{X'}
\end{tikzcd}$$

where $T'(\langle y|)(x) = \langle y, Tx \rangle_Y$

for $x \in D(T)$ $\quad = \langle T^* y, x \rangle_Y$

Riesz representation theorem: $X' \cong X$

(for Hilbert spaces)

antilinear
isometric
isomorphism

$C_X : X \to X', \quad x \mapsto \langle x, \cdot \rangle_X = \langle x|$

$C_Y : Y \to Y', \quad y \mapsto \langle y, \cdot \rangle_Y = \langle y|$

$$C_X^{-1} T' C_Y(y) = C_X^{-1} T'(\langle y|) \quad \text{for } y \in D(T^*)$$

$$= C_X^{-1}(\langle T^* y|)$$

$$= T^* y$$

$$\Longrightarrow \quad T^* = C_X^{-1} T' C_Y$$

**Proposition:** $X, Y$ Banach spaces, $T: X \supseteq D(T) \to Y$ densely defined operator.

Then: $T \subseteq S \implies T' \supseteq S'$

$\left( \begin{array}{l} D(T) \subseteq D(S), \ S \text{ extension of } T \\ \qquad Sx = Tx \ \text{ for all } x \in D(T) \end{array} \right)$ $\left( \begin{array}{l} D(T') \supseteq D(S'), \ S' \text{ restriction of } T' \\ \qquad S'y' = T'y' \ \text{ for all } y' \in D(S') \end{array} \right)$

And for Hilbert spaces: $T \subseteq S \implies T^* \supseteq S^*$

**Proof:** $D(S') := \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Sx) = x'(x) \text{ for all } x \in D(S) \}$

$$\subseteq \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in D(T) \}$$

$$= D(T') \qquad \square$$

# Unbounded Operators - Part 10

$T: X \supseteq D(T) \to Y$ densely defined $\Longrightarrow$ adjoint exists:

($X, Y$ Banach spaces) $T': Y' \supseteq D(T') \to X'$

($X, Y$ Hilbert spaces) $T^*: Y \supseteq D(T^*) \to X$

Definition: Let $X = L^2(\mathbb{R}, \mathbb{C})$ ← square-integrable functions with respect to the one-dimensional Lebesgue measure $\int_{\mathbb{R}} |f(x)|^2 \, dx < \infty$

Hilbert space with inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) \, dx$$

Let $\varphi: \mathbb{R} \to \mathbb{C}$ be a continuous function.

Then $M_\varphi : X \supseteq D(M_\varphi) \to X$ denotes the multiplication operator:

$f \mapsto M_\varphi f$ with $(M_\varphi f)(x) = \varphi(x) f(x)$

for $x \in \mathbb{R}$ almost everywhere

$$D(M_\varphi) := \{ f \in L^2(\mathbb{R}, \mathbb{C}) \mid \varphi \cdot f \in L^2(\mathbb{R}, \mathbb{C}) \}$$

dense in $L^2(\mathbb{R}, \mathbb{C})$

Adjoint of the multiplication operator: $(M_\varphi)^* : X \supseteq D((M_\varphi)^*) \to X$

$\{ g \in X \mid$ there is $\tilde{f} \in X$ with $\langle g, M_\varphi f \rangle = \langle \tilde{f}, f \rangle$ for all $f \in D(M_\varphi) \}$ // with $(M_\varphi)^* g = \tilde{f}$

## Is it a multiplication operator as well?

$$\langle g, M_\varphi f\rangle = \int_{\mathbb{R}} \overline{g(x)}\,\varphi(x) f(x)\,dx = \int_{\mathbb{R}} \overline{\overline{\varphi(x)}\, g(x)}\, f(x)\,dx = \langle M_{\overline{\varphi}}\, g, f\rangle$$

for all $f, g \in D(M_\varphi) = D(M_{\overline{\varphi}})$

First result: $M_{\overline{\varphi}} \subseteq (M_\varphi)^*$

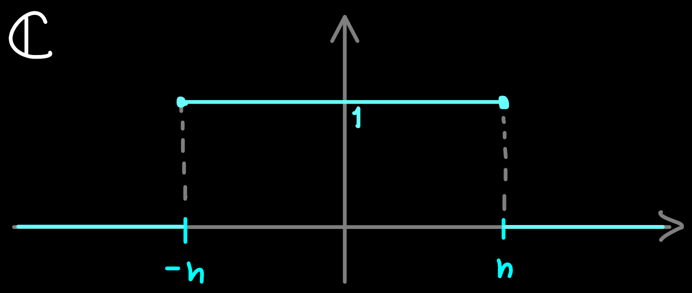
To show: $g \in D((M_\varphi)^*) \implies \overline{\varphi}\cdot g \in L^2(\mathbb{R}, \mathbb{C})$

Proof: Note: $g \in L^2$, $h$ bounded $\implies h \cdot g \in L^2$

Make $\overline{\varphi}$ bounded? Take $\chi_n : \mathbb{R} \to \mathbb{C}$

$\chi_n \overline{\varphi}$ is bounded

$(\chi_n \overline{\varphi})(x) \xrightarrow{n\to\infty} \overline{\varphi}(x)$ for $x \in \mathbb{R}$




For $f \in D(M_\varphi)$, $g \in D((M_\varphi)^*)$ :

$$\langle \chi_n (M_\varphi)^* g, f\rangle = \int_{\mathbb{R}} \overline{\chi_n(x)\, (M_\varphi)^* g(x)}\, f(x)\,dx$$

$$= \langle (M_\varphi)^* g, \chi_n f\rangle = \langle g, M_\varphi(\chi_n f)\rangle$$

$$= \int_{\mathbb{R}} \overline{g(x)}\, \varphi(x)\, \chi_n(x)\, f(x)\,dx$$

$$= \int_{\mathbb{R}} \overline{\overline{\varphi(x)}\, \chi_n(x)\, g(x)}\, f(x)\,dx = \langle \chi_n \overline{\varphi} g, f\rangle$$

$D(M_\varphi)$ dense $\implies \chi_n (M_\varphi)^* g = \chi_n \overline{\varphi} g \overset{\chi_n \xrightarrow{n\to\infty} 1}{\implies} (M_\varphi)^* g = \overline{\varphi} g \in L^2$

Final result: $(M_\varphi)^* = M_{\overline{\varphi}}$