Unbounded Operators – Part 8

For bounded operators: $T: X \to Y$ (Hilbert spaces) $\rightsquigarrow$ $T^*: Y \to X$ adjoint

$$\langle y, Tx \rangle_Y = \langle T^* y, x \rangle_X$$

$T: X \to Y$ (Banach spaces) $\rightsquigarrow$ $T': Y' \to X'$ adjoint

$$T'(y')(x) = y'(Tx)$$

for $y' \in Y'$, $x \in X$

Proposition: X, Y Banach spaces, $T: X \supseteq D(T) \to Y$ densely defined operator ($\hookrightarrow \overline{D(T)} = X$)

Then there is an operator $T': Y' \supseteq D(T') \to X'$ with

$$y'(Tx) = T'(y')(x) \quad \text{for } x \in D(T),\ y' \in D(T').$$

The domain $D(T')$ can be chosen maximally.

Proof: set $D(T') := \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in D(T) \}$

and define: $T'(y') := x'$

Well-defined? Assume there are $x_1', x_2' \in X'$ with $y'(Tx) = x_1'(x)$, $y'(Tx) = x_2'(x)$ for all $x \in D(T)$

$\Rightarrow x_1'(x) = x_2'(x)$ for all $x \in D(T)$

$\Rightarrow (x_1' - x_2')(x) = 0$ for all $x \in D(T)$ $\overset{\text{dense}}{\underset{\text{continuity}}{\Longrightarrow}}$ $(x_1' - x_2')(x) = 0$ for all $x \in X$

$\Rightarrow x_1' = x_2'$ $\square$

For Hilbert spaces: X, Y Hilbert spaces, $T: X \supseteq D(T) \to Y$ densely defined operator ($\hookrightarrow \overline{D(T)} = X$)

$$D(T^*) := \{ y \in Y \mid \text{there is } \tilde{x} \in X \text{ with } \langle y, Tx \rangle_Y = \langle \tilde{x}, x \rangle_X \text{ for all } x \in D(T) \}$$

$$T^*(y) := \tilde{x}$$