Weierstrass M-test

⇝ comparison test for uniform convergence

Theorem: D set, $f_k : D \to \mathbb{C}$ functions for each $k \in \mathbb{N}$.

If there are constants $M_k \geq 0$ such that $\sum_{k=1}^{\infty} M_k < \infty$ and

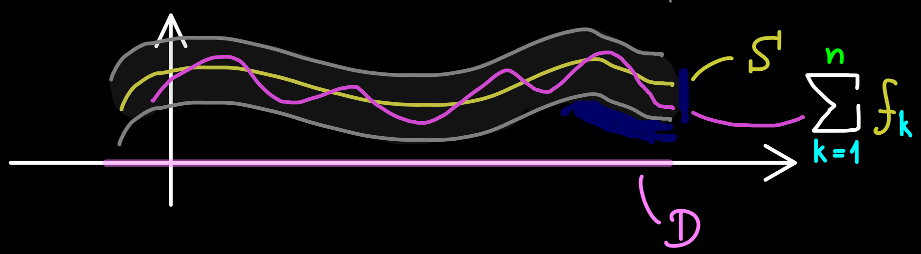
$|f_k(x)| \leq M_k$ for all $k \in \mathbb{N}$, $x \in D$,

then: $\sum_{k=1}^{\infty} f_k$ is uniformly convergent $\left(\sum_{k=1}^{\infty} |f_k| \text{ is uniformly convergent} \right)$

More precisely: $S : D \to \mathbb{C}$, $x \mapsto \sum_{k=1}^{\infty} f_k(x)$ is well-defined

and $\left\| S - \sum_{k=1}^{n} f_k \right\|_\infty \xrightarrow{n \to \infty} 0$

supremum norm



Proof: Cauchy criterion: $\sum_{k=1}^{\infty} M_k$ convergent $\Longleftrightarrow$ $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq m \geq N :$

$$\left| \sum_{k=m}^{n} M_k \right| < \varepsilon$$

For a given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ as above.

For every $x \in D$:

$$\left| \sum_{k=m}^{n} f_k(x) \right| \overset{\Delta\text{-inequality}}{\leq} \sum_{k=m}^{n} |f_k(x)| \leq \sum_{k=m}^{n} M_k < \varepsilon$$

$S : D \to \mathbb{C}$, $x \mapsto \sum_{k=1}^{\infty} f_k(x)$ exists!

Hence: For a given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$, $n \geq N$ as above.

$$\left\| S - \sum_{k=1}^{n} f_k \right\|_\infty = \sup_{x \in D} \left| S(x) - \sum_{k=1}^{n} f_k(x) \right| = \sup_{x \in D} \left| \lim_{m \to \infty} \sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x) \right|$$

$$= \sup_{x \in D} \lim_{m \to \infty} \left| \sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| = \sup_{x \in D} \lim_{m \to \infty} \underbrace{\left| \sum_{k=n+1}^{m} f_k(x) \right|}_{< \varepsilon}$$

$$\leq \varepsilon$$

$$\Longrightarrow \left\| S - \sum_{k=1}^{n} f_k \right\|_\infty \xrightarrow{n \to \infty} 0 \qquad \square$$

Example: $f_k : \mathbb{R} \to \mathbb{R}$, $f_k(x) = \dfrac{\cos(kx)}{k^2}$, $|f_k(x)| \leq \dfrac{1}{k^2}$ for all $x \in \mathbb{R}$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \xRightarrow{\text{Weierstrass M-test}} \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} \text{ converges uniformly}$$