Problem 1 *Important inequalities and properties of the absolute value* (4 points)
Prove the following statements:

a) $\forall a, b \in \mathbb{R} : |ab| \leq \frac{1}{2}\left(a^2 + b^2\right)$.

b) $\forall a, b \in \mathbb{R} : \big| |a| - |b| \big| \leq |a - b|$.

c) $\forall a, b \in \mathbb{R} : |ab| = |a| \cdot |b|$.

d) $\forall n \in \mathbb{N}\ \forall h \in \mathbb{R} : h \geq -1 \Rightarrow (1 + h)^n \geq 1 + hn$.
Hint: Use Induction.

Solutions:

(a) Claim: $|ab| \leq \frac{1}{2}(a^2 + b^2)$

Proof: $(a \pm b)^2 \geq 0$ (see problem 1.1. (d))

$\Rightarrow a^2 \pm 2ab + b^2 \geq 0$

$\Rightarrow \frac{1}{2}(a^2 + b^2) \geq \mp ab$

Since $|ab| = \begin{cases} +ab, & ab > 0 \\ 0, & ab = 0 \\ -ab, & ab < 0 \end{cases}$, we have $\frac{1}{2}(a^2+b^2) \geq |ab|$.

(b) Claim: $\big| |a| - |b| \big| \leq |a-b|$

Proof: From the lecture, we know the Δ-inequality:

$$|\tilde{a} \pm \tilde{b}| \leq |\tilde{a}| + |\tilde{b}|.$$ Hence for setting $\tilde{a} = a-b$, $\tilde{b} = b$:

$$|a| \leq |a-b| + |b| \quad \Rightarrow \quad |a| - |b| \leq |a-b|.$$

Analogously we can set $\tilde{a} = a$, $\tilde{b} = a-b$

$$|b| \leq |a-b| + |a| \quad \Rightarrow \quad |b| - |a| \leq |a-b|$$

In summary, we have:

$$\big| |a| - |b| \big| \leq |a-b|$$

(c) Claim: $|ab| = |a| \cdot |b|$

Proof: First case: $a \geq 0$, $b \geq 0$:

$$|ab| = a \cdot b = |a| \cdot |b| \quad \checkmark$$

Second case: $a \geq 0$, $b < 0$:

$$|a \cdot b| = |a(-b) \cdot (-1)| = |\underbrace{\underbrace{(-1)}_{<0} \cdot \underbrace{a(-b)}_{>0}}_{<0 \text{ (cf. 1.1(c))}}| = -(-1)(a)(-b)$$

$$= a \cdot (-b)$$

$$= |a| \cdot |b| \quad \checkmark$$

Third case: $a < 0$, $b < 0$:

$$|a \cdot b| = |\underbrace{\underbrace{(-1)^2}_{>0} \underbrace{(-a)}_{>0} \underbrace{(-b)}_{>0}}_{>0}| = (-a)(-b) = |a| \cdot |b| \quad \checkmark \quad \square$$

(d) Claim: $h \geq -1 \Rightarrow (1+h)^n \geq 1+hn$ for all n

Proof: Ind. start: $n_0 = 1$: $(1+h)^1 = 1+h \geq 1 + 1 \cdot h$

Ind. step: $n \to n+1$: $(1+h)^{n+1} = (1+h)(1+h)^n$

$$\underset{\text{(ind. hypo)}}{\geq} \underbrace{(1+h)}_{\geq 0}(1+nh)$$

$$\Rightarrow (1+h)^{n+1} \geq 1+h+nh+nh^2 = 1+(n+1)h+nh^2$$

$$\geq 1+(n+1)h \quad \checkmark$$

$\square$