Problem 3 *Recursively defined sequences* (3 points)
We consider a recursively defined sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ given by

$$a_1 := 1, \quad a_{n+1} := \sqrt{a_n + 1}.$$

Prove that (a_n) is convergent using the monotonicity criterion and find its limit.

Solutions

$a_1 = 1, \quad a_{n+1} = \sqrt{a_n + 1}$ (Square root is monot. increasing!)

Claim: a_n is bounded by 3.

Proof: Induction: $a_1 = 1 \leq 3$.

Step $n \to n+1$: $a_{n+1} = \sqrt{a_n + 1} \overset{(IH)}{\leq} \sqrt{3+1} = 2 \leq 3$.

Claim: a_n is monotonically increasing

Proof: Induction: $a_2 = \sqrt{a_1 + 1} = \sqrt{2} > 1 = a_1$ ✓

Step $n \to n+1$: $a_{n+2} = \sqrt{a_{n+1} + 1} \overset{(IH)}{\geq} \sqrt{a_n + 1} = a_{n+1}$.

By video 7, we have the fact that every monotonically increasing sequence that is also bounded from above has to be a convergent sequence.

Let $a := \lim_{n\to\infty} a_n$ be the limit, then we have:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 1} \quad (\text{square root continuous!})$$

$$\Leftrightarrow \quad a = \sqrt{a+1} \quad \Rightarrow \quad a^2 - a - 1 = 0 \quad \Rightarrow a = \frac{1 \pm \sqrt{5}}{2}$$

a has to be positive $\Rightarrow \quad a = \dfrac{1+\sqrt{5}}{2}$