Problem 2 *Product and quotient of convergent sequences* (4 points)
Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be convergent sequences of real numbers and denote the limits by a, b and c, respectively. Prove the following:

a) $\exists M > 0\ \forall n \in \mathbb{N} : |a_n| \leq M$ (*a convergent sequence is also bounded*).

b) $a_n \cdot b_n \xrightarrow{n\to\infty} a \cdot b$.

c) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \frac{a}{b}$.

d) If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $c = a$, then we have $b = a$ (*Sandwich theorem*).

Solutions:

(a) Claim: There is an $M > 0$ such that for all $n \in \mathbb{N}$: $|a_n| \leq M$

Proof: $a \in \mathbb{R}$ is the limit of a_n. This means that for given $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N_\varepsilon.$$

(Choose $\varepsilon = 1$)

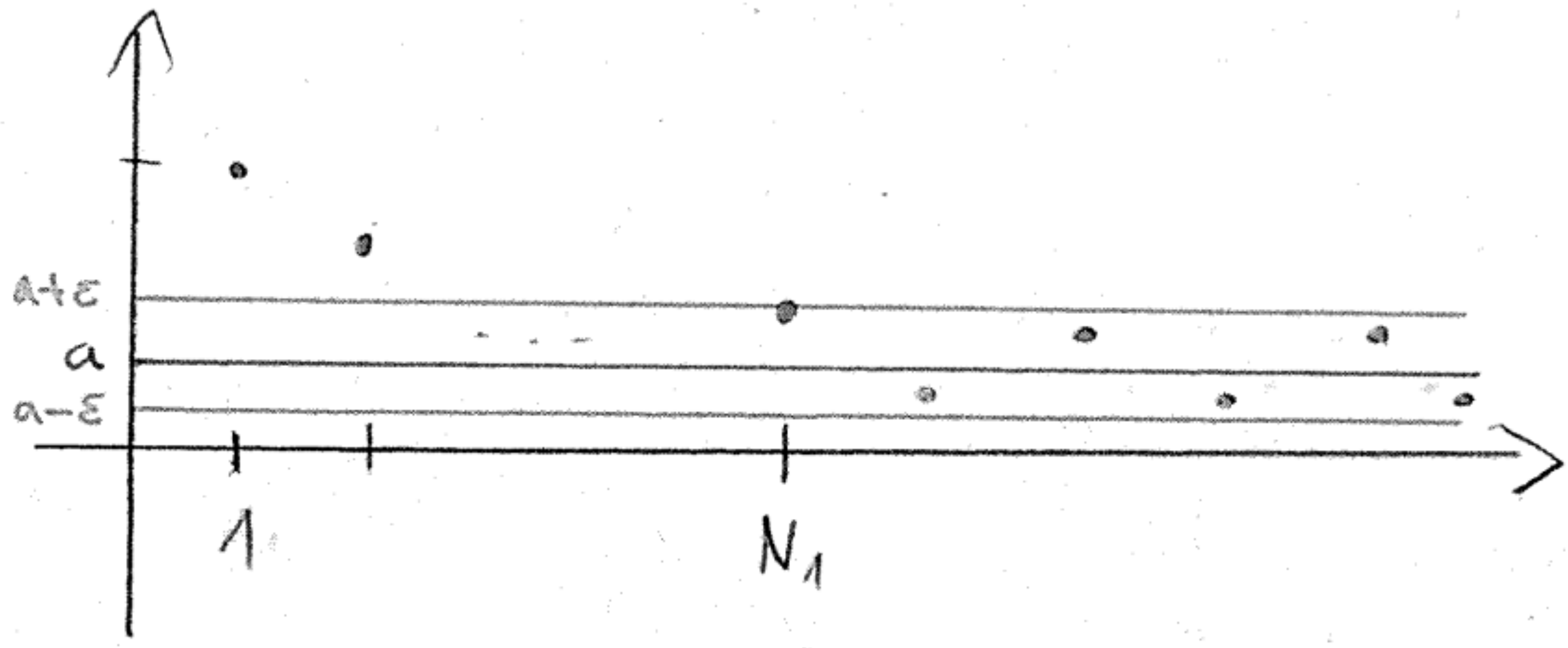
Set $M := |a_1| + \dots + |a_{N_1}| + 1 + |a| < \infty$

Then we have $|a_n| \leq M$ for all $n \leq N_1$.

For all $n \geq N_1$ we get:

$$|a_n| \leq \underbrace{|a_n - a|}_{<1\,(=\varepsilon)} + |a| \qquad (\Delta\text{-inequality})$$

$$\leq 1 + |a| \leq M.$$



1.2 (b)

Claim: $(a_n \cdot b_n)_{n\in\mathbb{N}}$ has limit $a\cdot b$.

Proof: Let $\varepsilon > 0$. Then we know, there is an $N_a \in \mathbb{N}$ and an $N_b \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N_a$$

$$|b_n - a| < \varepsilon \quad \text{for all } n \geq N_b$$

By part (a), there is an $M > 0$ for the sequence $(b_n)_{n\in\mathbb{N}}$ (meaning: $|b_n| \leq M$ for all $n \in \mathbb{N}$).

Then we get for all $n \geq \max\{N_a, N_b\} =: N$

$$|a_n b_n - ab| \leq |a_n b_n - ab_n + ab_n - ab|$$

$$\leq \underbrace{|b_n|}_{\leq M} \cdot \underbrace{|a_n - a|}_{<\varepsilon} + |a|\,\underbrace{|b_n - b|}_{<\varepsilon}$$

$$< M\cdot\varepsilon + |a|\cdot\varepsilon = (M+|a|)\cdot\varepsilon$$

This means that we just have to rescale:
Given $\varepsilon' > 0$ define $\varepsilon := \frac{1}{(M + |a|)} \cdot \varepsilon'$ and do the calculation from above. Then:

$$|a_n b_n - ab| < \varepsilon' \quad \text{for all } n \geq N.$$

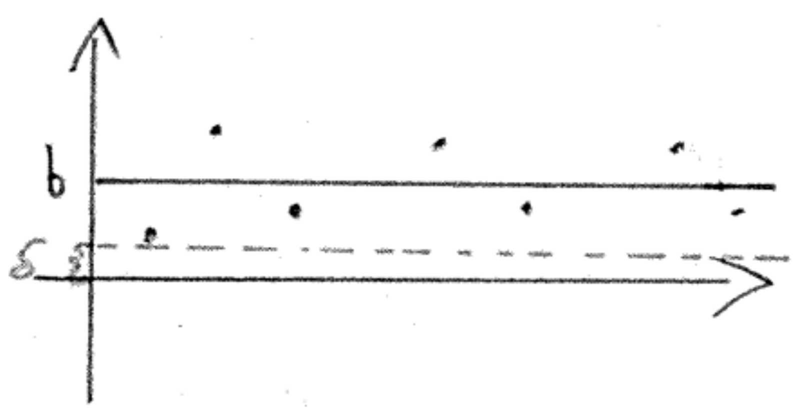
$\Rightarrow (a_n b_n)_{n \in \mathbb{N}}$ is convergent with limit $a \cdot b$. $\square$

1.2 (c)

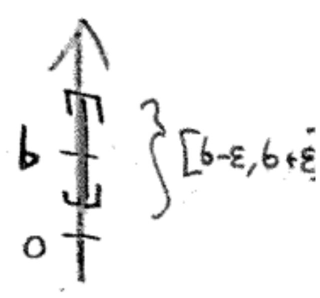
Claim: $\frac{a_n}{b_n} \xrightarrow{n \to \infty} \frac{a}{b}$

Proof: Since $b_n \neq 0$ and $b \neq 0$, there is a $\delta > 0$ such that

$$|b_n| \geq \delta \quad \text{for all } n \in \mathbb{N}.$$



To show this, we use that (b_n) converges to b and choose $\varepsilon = \frac{|b|}{2}$. Then there is an $\tilde{N} \in \mathbb{N}$ such that



$$b_n \in [b-\varepsilon,\ b+\varepsilon] = \left[\tfrac{1}{2}b,\ \tfrac{3}{2}b\right] \quad (\text{for } b>0)$$

for all $n \geq \tilde{N}$.

Choose $\delta := \min\left\{ |b_1|, \dots, |b_N|, \frac{|b|}{2} \right\}$.

Now we can show that $\left(\frac{1}{b_n}\right)_{n\in\mathbb{N}}$ converges to $\frac{1}{b}$.

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$

$$|b_n - b| < \delta \cdot |b| \cdot \varepsilon$$

Then for $n \geq N$:

$$\left| \frac{1}{d_n} - \frac{1}{b} \right| = \left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b_n \cdot b} \right|$$

$$= \frac{1}{|b_n \cdot b|} \cdot |b_n - b| \leq \frac{1}{\delta \cdot |b|} \cdot |b_n - b| < \varepsilon.$$

Therefore: $\frac{1}{b_n} \xrightarrow{n\to\infty} \frac{1}{b}$.

Combining this with part (b), we get $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \frac{a}{b}$.

1.2 (d)

Claim: $a_n \leq b_n \leq c_n$ and $a = c$ $\Rightarrow$ $b = a = c$

Proof: By using the limit theorem $\left(\lim\limits_{n\to\infty} (x_n + y_n) \overset{(*)}{=} \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n \right)$ we show that $(c_n - b_n)_{n\in\mathbb{N}}$ is a zero sequence:

$$0 \leq c_n - b_n \leq c_n - a_n \xrightarrow[(*)]{n\to\infty} c - a = 0$$

$$b_n = c_n - (c_n - b_n) \xrightarrow[(*)]{} c - 0$$

$\Rightarrow$ $(b_n)_{n\in\mathbb{N}}$ converges with limit $b = c = a$.