Problem 1 *Ordering* (4 points)
For $x, y \in \mathbb{R}$, we write the symbol $x \leq y$ for $(x < y \vee x = y)$. Use the axioms of $\mathbb{R}$ and show the following: If for a given $a \in \mathbb{R}$ the inequality

$$0 \leq a \leq \varepsilon$$

holds for every $\varepsilon > 0$, then $a = 0$.

Solutions

Claim: $\forall \varepsilon > 0 : \quad 0 \leq a \leq \varepsilon \implies a = 0$

Proof: First note that from $1 > 0$ and (O3) follows

$$2 = 1 + 1 > 1 > 0 \implies 2 > 0 \underset{\text{(Problem 1.1 (b))}}{\implies} \tfrac{1}{2} > 0 .$$

This means that $\frac{1}{2} b > 0$ for all $b > 0$. (*)

Now we do a proof by contradiction. Therefore, we assume

$0 \leq a \leq \varepsilon$ for all $\varepsilon > 0$ <u>and</u> $a \neq 0$. (hence: $a > 0$)

By using (*), we then also know $a \leq \frac{1}{2} a$.

Therefore: $2a \leq a$ ((O4) with $2 > 0$)

$\iff (1+1) a \leq a$

$\iff a + a \leq a$ (Distributivity)

$\iff a \leq 0$ ((O3) with $-a$)

$\iff a < 0$ or $a = 0$

This is a contradiction to $a > 0$ (and axiom (O1)) $\square$