Problem 2 *Binomial theorem and geometric sum formula* (4 points)
For $n, k \in \mathbb{N}_0$ with $n \geq k$, the binomial coefficient is defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Show for all $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$:

a) $(a+b)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

b) $a^n - b^n = (a-b) \displaystyle\sum_{k=0}^{n-1} a^k b^{n-1-k}$.

Solutions

(a) Claim: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Proof: Using induction: $n=0$: $(a+b)^0 = 1$, $\sum_{k=0} \binom{n}{k} a^k b^{n-k} = \underbrace{\binom{0}{0}}_{1} \underbrace{a^0}_{1} \cdot \underbrace{b^0}_{1} = 1$

induction step: $n \to n+1$:

$$(a+b)^{n+1} = (a+b)(a+b)^n \overset{\text{induction hypothesis}}{=} (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}$$

$$= \binom{n}{n}a^{n+1}b^0 + \sum_{l=1}^{n}\binom{n}{l-1}a^l b^{n-(l-1)} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k+1} + \binom{n}{0}a^0 b^{n+1}$$

$$= (*)$$

We shifted the index by $l = k+1$ to get the same formulas inside the sums.

Now we use Pascal's triangle formula:

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1} \quad , \text{ for } m \leq n \quad (P)$$

Then we get: (write m for the new sum index)

$$(*) = \binom{n}{n} a^{n+1} b^0 + \sum_{m=1}^{n} \left(\binom{n}{m-1} + \binom{n}{m} \right) a^m b^{n-m+1} + \binom{n}{0} a^0 b^{n+1}$$

$$\overset{(!)}{=} \binom{n+1}{n+1} a^{n+1} b^0 + \sum_{m=1}^{n} \binom{n+1}{m} a^m b^{n-m+1} + \binom{n+1}{0} a^0 b^{n+1} \qquad \left(\binom{n+1}{n+1} = 1 \right)$$

$$= \sum_{m=0}^{n+1} \binom{n+1}{m} a^m b^{(n+1)-m}.$$

This is the wanted formula for $n+1$.

(b) Claim: $a^n - b^n = (a-b) \sum_{k=0}^{n-} a^k b^{n-1-k}$

Proof: Again induction.

$n=0$: $\quad 0 = a^0 - b^0 = (a-b) \cdot 0 = (a-b) \cdot \sum_{k=0}^{-1} a^k b^{n-1-k}$

empty sum!

induction step: $n \to n+1$:

$$(a-b) \cdot \sum_{k=0}^{(n+1)-1} a^k b^{(n+1)-1-k} = (a-b) \sum_{k=0}^{n} a^k b^{n-k}$$

$$= \sum_{k=0}^{n} a^{k+1} b^{n-k} - \sum_{k=0}^{n} a^k b^{n+1-k}$$

$$= a^{n+1} b^{n-n} + (ab) \sum_{k=0}^{n-1} a^k b^{n-1-k} - a^n b^{n+1-n}$$
$$- b^2 \sum_{k=0}^{n-1} a^k b^{n-1-k}$$

$$= a^{n+1} + b \cdot (a-b) \cdot \sum_{k=0}^{n-1} a^k b^{n-1-k} - a^n b$$

$$\overset{(IH)}{=} a^{n+1} + b \cdot (a^n - b^n) - a^n b = a^{n+1} - b^{n+1} \quad \checkmark$$