Problem 3 *Null sequence* (4 points)
Show that the sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers given by

$$a_n = \sqrt[n]{n} - 1$$

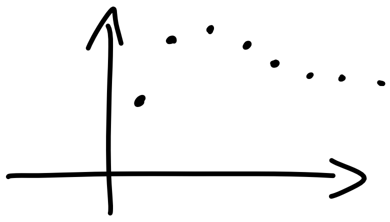
is convergent with limit 0.

Hint: The binomial theorem can be helpful.

Solutions

Claim: (a_n) converges with limit 0

Proof: Clearly: $n \geq 1 \Rightarrow \sqrt[n]{n} \geq 1$



Therefore $a_n \geq 0$ for all $n \in \mathbb{N}$.

We use the binomial theorem for $\sqrt[n]{n} = a_n + 1$:

$$n = (a_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k \qquad \text{(each summand is positive!)}$$

$$= 1 \cdot a_n^0 + n \cdot a_n^1 + \frac{n(n-1)}{2} a_n^2 + \dots$$

The sum is greater than one term and we choose the third !

Therefore: $n \geq \frac{n(n-1)}{2} a_n^2 \Rightarrow a_n^2 \leq \frac{2}{n-1} \xrightarrow{n\to\infty} 0$

Hence $(a_n)_{n\in\mathbb{N}}$ converges to 0. $\square$