**Problem 1** *Periodic subsequences* (4 points)
For each real number $a \in \mathbb{R}$, we denote its *integer part* by $\lfloor a \rfloor \in \mathbb{Z}$, which means that $\lfloor a \rfloor$ is the greatest integer such that $\lfloor a \rfloor \leq a$ holds. Furthermore, we define the fractional part $\mathrm{frac}(a) := a - \lfloor a \rfloor \in [0, 1)$. For each $\alpha \in \mathbb{R}$, we consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n := \mathrm{frac}(\alpha n)$.

a) Prove that $\alpha \in \mathbb{Q}$ if and only if $(a_n)_{n\in\mathbb{N}}$ is *periodic*, which means that there is a $p \in \mathbb{N}$ such that $a_{n+p} = a_n$ for all $n \in \mathbb{N}$.

b) For $\alpha = \frac{5}{3}$ find a subsequence of $(a_n)_{n\in\mathbb{N}}$ that converges.

c) For $\alpha = \frac{5}{3}$ determine the *period* of $(a_n)_{n\in\mathbb{N}}$, which is the smallest number $p$ with the property given in part a).

d) For $\alpha = \frac{5}{3}$ find all accumulation points (*also called limit points*) of $(a_n)_{n\in\mathbb{N}}$.

## Solutions

Define $a_n := \mathrm{frac}(\alpha n)$ with $\mathrm{frac}(a) := a - \lfloor a \rfloor$

(a) Claim: $\alpha \in \mathbb{Q} \iff (a_n)_{n\in\mathbb{N}}$ is periodic

Proof: $(a_n)_{n\in\mathbb{N}}$ is periodic $\iff$ There is a $p \in \mathbb{N}$ with $a_{n+p} = a_n$ for all $n$

$\iff \mathrm{frac}(\alpha(n+p)) = \mathrm{frac}(\alpha n)$ (for a $p \in \mathbb{N}$ and all $n \in \mathbb{N}$)

$\iff \underbrace{\alpha(n+p) - \alpha n}_{= \alpha \cdot p} = \underbrace{\lfloor \alpha(n+p) \rfloor - \lfloor \alpha n \rfloor}_{\in \mathbb{Z}}$ (for a $p \in \mathbb{N}$ and all $n \in \mathbb{N}$)

$\iff$ There is $p \in \mathbb{N}$ such that: $\alpha \cdot p \in \mathbb{Z}$

$\iff \alpha \in \mathbb{Q}$ $\square$

(b) Set $\alpha = \frac{5}{3}$. Then $\mathrm{frac}(\alpha 1) = \frac{2}{3}$, $\mathrm{frac}(\alpha \cdot 2) = \frac{1}{3}$

$$a_n = \mathrm{frac}\left(\tfrac{5}{3}\cdot n\right) = \tfrac{5}{3}\cdot n - \underbrace{\left\lfloor \tfrac{5}{3}\cdot n \right\rfloor}_{= \begin{cases} 5m, & n = 3m \text{ for a } m\in\mathbb{N} \\ 5m-2, & n = 3m-1 \text{ for a } m\in\mathbb{N} \\ 5m-4, & n = 3m-2 \text{ for a } m\in\mathbb{N} \end{cases}}$$

Therefore: $a_{3m} = 5\cdot m - 5m = \underline{0}$ for all $m \in \mathbb{N}$

or $a_{3m-1} = \frac{5}{3}\cdot(3m-1) - (5m-2) = 2 - \frac{5}{3} = \underline{\frac{1}{3}}$

or $a_{3m-2} = \frac{5}{3}(3m-2) - (5m-4) = 4 - \frac{10}{3} = \underline{\frac{2}{3}}$

$\Rightarrow$ All subsequences are constant and therefore convergent!

(c) The proof of a shows that every $p \in \mathbb{N}$ with the property $a_{n+p} = a_n$ for all $n \in \mathbb{N}$ fulfils:

$$\alpha \cdot p \in \mathbb{Z}$$

The smallest natural number for $\alpha = \frac{5}{3}$ with this property is $p = 3$. Hence, the period is $\underline{3}$

(d)

In (b), we have written three subsequences that have given us three accumulation points:

$$0, \frac{1}{3}, \frac{2}{3}$$

Since the period for $(a_n)_{n \in \mathbb{N}}$ is 3, there can't be any more accumulation points since the sequence $(a_n)_{n \in \mathbb{N}}$ can only consists of these three numbers.