Problem 2 *Infimum, supremum, minimum and maximum* (4 points)
Determine the supremum and infimum and, in case of existence, the minimum and maximum of the following sets:

a) $A := \{3 + \frac{5}{8n^2} \mid n \in \mathbb{N}\}$,

b) $B := \{x \in \mathbb{R} \mid |x+2| - |x-3| \leq 1\}$,

c) $C := \left\{\frac{3n^2}{2^n} \mid n \in \mathbb{N}\right\}$,

d) $D := \left\{x \in \mathbb{R} \setminus \{-1\} \,\middle|\, \frac{|x-1|}{|x+1|} \geq 1\right\}$.

Solutions

(a) The sequence $3 + \frac{5}{8n^2}$ is mon. decreasing. Hence:

$$\sup A = 3 + \frac{5}{8 \cdot 1^2} = 3 + \frac{5}{8}$$

$$\inf A = \lim_{n \to \infty} \left(3 + \frac{5}{8n^2}\right) = 3$$

Since $\frac{5}{8n^2} > 0$ for all n, there is no minimum and the maximum is given by $3 + \frac{5}{8}$.

(b) For $x \leq -2$: $|x+2| - |x-3| = -(x+2) - (-(x-3))$
$= -x - 2 + x - 3 = -5 \leq 1$.

This means that $(-\infty, 2] \subseteq M \Rightarrow \inf(B)$ and $\min(B)$ don't exist in $\mathbb{R}$
(in other words: $\inf(B) = -\infty$)

For $x \in [-2,3]$, we find $|x+2| - |x-3| = (x+2) - (-(x-3))$

$$= x+2+x-3 = 2x-1$$

This means $x \in B \cap [-2,3] \iff 2x-1 \leq 1 \iff x \leq 1$

For $x > 3$, we find $|x+2| - |x-3| = (x+2) - (x-3) = 5 > 1$.

Therefore: $B = (-\infty, 1] \overset{B \text{ is interval}}{\Longrightarrow} \sup(B) = 1$

(c)

$\frac{3n^2}{2^n}$ is a null sequence. Show, for example:

$$2^n = (1+1)^n \overset{\text{binomial theorem}}{=} \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \underbrace{1}_{>0} + \underbrace{n}_{>0} + \underbrace{\frac{n(n-1)}{2}}_{\cdots} + \frac{n(n-1)(n-2)}{6} + \cdots$$

$$\geq \frac{n(n-1)(n-2)}{6}$$

$$\Rightarrow \quad 0 \leq \frac{3n^2}{2^n} \leq 3n^2 \cdot \frac{6}{n(n-1)(n-2)} = \frac{18 \cdot \frac{1}{n}}{1 \cdot (1-\frac{1}{n})(1-\frac{2}{n})}$$

$18 \cdot \frac{1}{n} \xrightarrow{n \to \infty} 0$, $\quad 1 \cdot (1-\frac{1}{n})(1-\frac{2}{n}) \xrightarrow{n \to \infty} 1$

Sandwich-Theorem

$$\Rightarrow \quad a_n := \frac{3n^2}{2^n} \xrightarrow{n \to \infty} 0$$

Therefore: $\inf(C) = 0$ but $\min(C)$ doesn't exist.

$\sup(C) = \max(C) = \underbrace{\frac{27}{8}}_{a_3}$ since $(a_n)_{n \in \mathbb{N}}$ is mon. decreasing for $n > 3$.

(d) First case: $x < -1$:

$$x \in D \cap (-\infty, -1) \iff \frac{-(x-1)}{-(x+1)} \geq 1$$

$$\iff x-1 \leq x+1 \iff -1 \leq 1$$

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Therefore: $(-\infty, -1) \subseteq D$.

Second case: $1 \geq x > -1$:

$$x \in D \cap (-1, 1] \iff \frac{-(x-1)}{x+1} \geq 1 \iff 1 \geq 2x+1$$

$$\iff x \leq 0$$

Therefore: $(-1, 0] \subseteq D$

Third case: $x \geq 1$:

$$x \in D \cap [1, \infty) \iff \frac{x-1}{x+1} \geq 1 \iff x-1 \geq x+1$$

$$\iff -1 \geq 1$$

In summary: $D = (-\infty, 0] \setminus \{-1\}$

Therefore:

$\sup D = 0$

$\inf D = -\infty$ (doesn't exist in $\mathbb{R}$)

$\max D = 0$

$\min D = -\infty$ (does not exist)