Problem 1 *Complex sequences* (4 points)
Show that the following sequences of complex numbers, denoted by $(z_n)_{n\in\mathbb{N}}$, are convergent and find their limits:

a) $z_n = \mathrm{e}^{\mathrm{i}\frac{9\pi}{n}} \cdot \frac{1}{n}$

b) $z_n = 3^{-n}\mathrm{e}^{\mathrm{i}n!}$

c) $z_n = \frac{1}{n}\sum_{k=0}^{n} \mathrm{i}^k$

d) $z_{n+1} = \frac{1}{2}\left(z_n - \frac{1}{z_n}\right)$, $z_1 = \mathrm{i}q_1$ where $q_1 \in \mathbb{R}$ with $q_1 > 0$.

Solutions

(a)

$$\left| \frac{1}{n} e^{i\frac{9\pi}{n}} \right| = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

so z_n also converges to 0.

(b)

$$z_n = 3^{-n} e^{in!} = 3^{-n}\left(\cos(n!) + i\sin(n!)\right)$$

$$\Rightarrow \quad 0 \leq 3^{-n}\cos(n!) \leq 3^{-n} \xrightarrow{n\to\infty} 0$$

Sandwich-Theorem says $a_n := 3^{-n}\cos(n!)$ is a null sequence.

Analogously: $b_n := 3^{-n}\sin(n!)$ is a null sequence.

Since $z_n = a_n + ib_n$, we have $\lim_{n\to\infty} z_n = 0$

(c) $z_n = \frac{1}{n}\sum_{k=0}^{n} i^k$, $z_1 = 1\cdot(1+i)$, $z_2 = \frac{1}{2}(1+i-1)$

$z_3 = \frac{1}{3}(1+i-1-i) = 0$

$z_4 = \frac{1}{4}(1+i-1-i+1) = \frac{1}{4}$

So we see:

$$z_n = \begin{cases} \frac{1}{n} & , \; n = 4m \quad \text{for an } m \in \mathbb{N} \\ 0 & , \; n = 4m-1 \quad \text{for an } m \in \mathbb{N} \\ \frac{i}{n} & , \; n = 4m-2 \quad \text{"} \\ \frac{1+i}{n} & , \; n = 4m-3 \quad \text{"} \end{cases}$$

Therefore <u>all</u> subsequences of z_n converge to 0.

$\Rightarrow \lim_{n\to\infty} z_n = 0$

(d) $z_1 = i q_1$ for $q_1 > 0$ and $z_{n+1} = \frac{1}{2}\left(z_n - \frac{1}{z_n}\right)$

$$\Rightarrow z_2 = \frac{1}{2}\left(i q_1 - \frac{1}{i q_1}\right) = \frac{1}{2}\left(\frac{-q_1^2 - 1}{i q_1}\right) = \frac{i}{2}\left(\frac{q_1^2+1}{q_1}\right)$$

$$= i \cdot q_2 \qquad \text{with} \quad q_2 = \frac{1}{2}\left(q_1 + \frac{1}{q_1}\right)$$

Then also $z_3 = i q_3$ with $q_3 = \frac{1}{2}\left(q_2 + \frac{1}{q_2}\right)$,

and so on.

Hence, we get a sequence of real numbers q_n defined by $q_{n+1} = \frac{1}{2}\left(q_n + \frac{1}{q_n}\right)$.

Redoing the proof from the tutorial, we get that (q_n) is monotonically decreasing, eventually, and bounded. Therefore (q_n) converges to limit

$$q := \lim_{n\to\infty} q_n \quad \text{where:}$$

$$q = \frac{1}{2}\left(q + \frac{1}{q}\right) \Rightarrow q = \pm 1 \overset{q_1 > 0}{\Rightarrow} \underline{q = 1}$$

This means that $(z_n)_{n\in\mathbb{N}}$ converges to $z = i$.