Problem 2 *Closedness* (4 points)
Let $X := \{z \in \mathbb{C} \mid \mathrm{Re}(z) \in [0,1)\,,\ \mathrm{Im}(z) \in [0,1]\}$ be a set in the complex numbers $\mathbb{C}$.

a) Show that the sequences $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$, given by
$$z_n = \frac{1}{n} + \mathrm{i}\left(1 - \frac{1}{n}\right), \quad w_n = 1 - \frac{1}{n} + \frac{\mathrm{i}}{n}, \quad n \geq 2$$
and
$$z_1 = \frac{1}{2} + \frac{\mathrm{i}}{2}, \quad w_1 = \frac{1}{2} + \frac{\mathrm{i}}{2},$$
are Cauchy sequences in X.

b) Are the sequences $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ convergent and do the limits lie in X?

c) Draw the set X and the sequence members of $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ in the complex plane.

d) What does one have to add to the set X, at least, such that we obtain a closed set?

Solutions

(a) (z_n) converges to $0 + i$
(w_n) converges to $1 + 0 \cdot i$

They are convergent in $\mathbb{C}$ and therefore also Cauchy sequences in $\mathbb{C}$.

Since all members z_n, w_n lie in X, the sequences are also Cauchy sequences in X.

(The Cauchy property just needs the members, not the limit itself)

(b) Both are convergent sequences in $\mathbb{C}$.
But also convergent in the smaller set X?

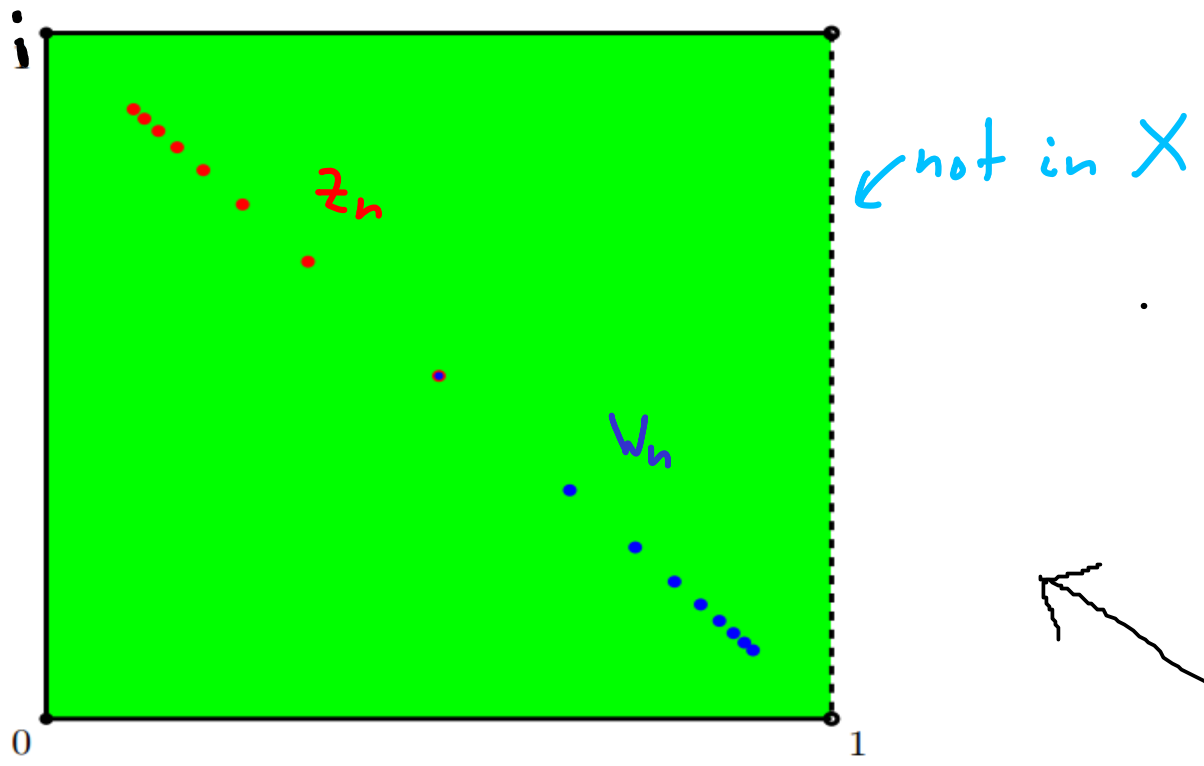
The limit of (z_n) lies in X $\Rightarrow$ convergent in X

The limit of (w_n) does not lie in X

$\Rightarrow$ not convergent in X

(X is not complete like $\mathbb{C}$ or $\mathbb{R}$!)

(c)



(d) We have to add all limits of Cauchy sequences in X!
This means we have to add the line on the right-hand side to get a closed set.