Problem 2 *Limit superior and limit inferior* (4 points)
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Use the definitions of the limit superior and the limit inferior from the tutorial and prove the following statements:

a) If $(a_n)_{n\in\mathbb{N}}$ is bounded, then $\liminf_{n\to\infty} a_n$ is the smallest and $\limsup_{n\to\infty} a_n$ the biggest accumulation point of $(a_n)_{n\in\mathbb{N}}$.

b) The sequence $(a_n)_{n\in\mathbb{N}}$ is convergent if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \notin \{\pm\infty\}$.

c) The sequence $(a_n)_{n\in\mathbb{N}}$ is divergent to ∞ if and only if $\liminf_{n\to\infty} a_n = \infty$. In this case, we have $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$.

d) Now let the sequence given by $a_n := 2^n(1+(-1)^n)+1$ for $n\in\mathbb{N}$. Determine

$$\liminf_{n\to\infty} a_n\,, \qquad \limsup_{n\to\infty} a_n\,, \qquad \liminf_{n\to\infty} \frac{a_{n+1}}{a_n}\,,$$
$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}\,, \qquad \liminf_{n\to\infty} \sqrt[n]{a_n}\,, \qquad \limsup_{n\to\infty} \sqrt[n]{a_n}\,.$$

Solutions

(a) Claim: $\liminf_{n\to\infty} a_n$ smallest accumulation point.

Proof: Of course, $\liminf a_n$ is an accumulation point. Just choose a subsequence $(a_{n_j})_{j\in\mathbb{N}}$ by choosing

$$a_{n_j} < \inf\{a_k \mid k\geq j\} + \frac{1}{j}\,.$$

Then:

$$\inf\{a_k \mid k \geq j\} \leq a_{n_j} \leq \inf\{a_k \mid k \geq j\} + \frac{1}{j} .$$

$j \leq n_j$

Sandwich theorem says:

$$\lim_{j\to\infty} \inf\{a_k \mid k \geq j\} \leq \lim_{j\to\infty} a_{n_j} \leq \lim_{j\to\infty} \inf\{a_k \mid k \geq j\}$$

$$\Rightarrow \lim_{j\to\infty} a_{n_j} = \liminf_{n\to\infty} a_n .$$

It has to be the smallest one, since every subsequence $(a_{n_\ell})_{\ell\in\mathbb{N}}$ is bounded from below by $\inf\{a_k \mid k \geq \ell\}$.

The supremum proof is analogous!

(Or just substitute $b_n := -a_n$)

(b) Claim: $(a_n)_{n\in\mathbb{N}}$ convergent $\Leftrightarrow$ $\liminf = \limsup \in \mathbb{R}$

Proof: (a_n) convergent with limit $a \in \mathbb{R}$

$$\Leftrightarrow \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N} \quad |a_n - a| < \varepsilon$$

$$\Leftrightarrow \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N} \quad a-\varepsilon \leq a_n \leq a+\varepsilon$$

$$\Leftrightarrow \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N}$$

$$a-\varepsilon \leq \inf\{a_k \mid k\geq n\} \leq a_n \leq \sup\{a_k \mid k \geq n\} \leq a+\varepsilon$$

$$\Leftrightarrow \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N}$$

$$\left|\inf\{a_k \mid k\geq n\} - a\right| < \varepsilon \text{ and } \left|\sup\{a_k \mid k\geq n\} - a\right| < \varepsilon$$

$$\Leftrightarrow \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R} \qquad \square$$

Alternatively: (a_n) convergent with limit $a \in \mathbb{R}$

$\Leftrightarrow$ (a_n) bounded and only one accumulation point (a.p.)

$\Leftrightarrow$ (a_n) bounded and biggest a.p. = smallest a.p.

$\overset{(a)}{\Leftrightarrow}$ $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ and both finite! $\square$

(c) <u>Claim:</u> $\lim\limits_{n\to\infty} a_n = \infty \iff \liminf a_n = \infty$

<u>Proof:</u> $(a_n)_{n\in\mathbb{N}}$ divergent to ∞

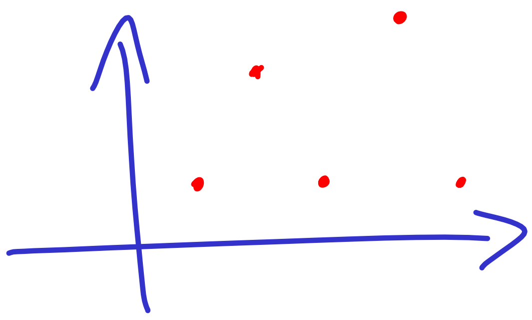
$$\iff \forall_{C>0} \ \exists_{N\in\mathbb{N}} \ \forall_{n\geq N} \quad |a_n| \geq c$$

$$\iff \forall_{C>0} \ \exists_{N\in\mathbb{N}} \ \forall_{n\geq N} \quad \inf\{a_k \mid k\geq n\} \geq c$$

$\iff$ The sequence $(i_n)_{n\in\mathbb{N}}$ with $i_n := \inf\{a_k \mid k \geq n\}$ is divergent to ∞

$$\iff \liminf_{n\to\infty} a_n = \infty$$

(d) $$a_n = 2^n(1+(-1)^n)+1$$



$$\inf\{a_k \mid k\geq n\} = 1$$

$$\sup\{a_k \mid k \geq n\} = \infty \qquad \text{(unbounded)}$$

$$\Rightarrow \liminf_{n\to\infty} a_n = 1 \quad \text{und} \quad \limsup_{n\to\infty} a_n = \infty$$

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1}{2^{n+1}+1}, & n \text{ even} \\ 2^{n+2}+1, & n \text{ odd} \end{cases}$$

$$\Rightarrow \quad \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$$

and $$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$$

$$\sqrt[n]{a_n} = \begin{cases} \sqrt[n]{1} & , \quad n \text{ odd} \\ \underline{\sqrt[n]{2^{n+1}+1}} & , \quad n \text{ even} \end{cases}$$

Since $\underline{2 \cdot \sqrt[n]{4}} = \sqrt[n]{2^{n+1} \cdot 2} \geq \underline{\sqrt[n]{2^{n+1}+1}} \geq \sqrt[n]{2 \cdot 2^n} = \underline{2 \cdot \sqrt[n]{2}}$,

we have by the sandwich theorem:

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = 2$$

and $$\liminf_{n\to\infty} \sqrt[n]{a_n} = 1$$