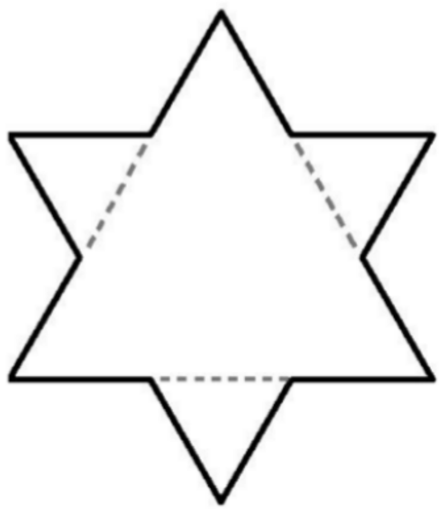
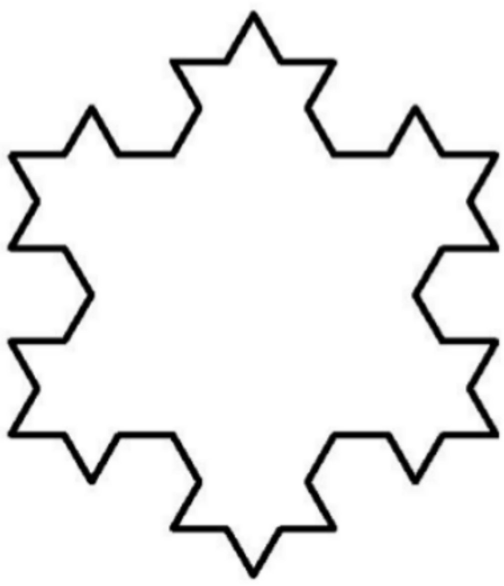
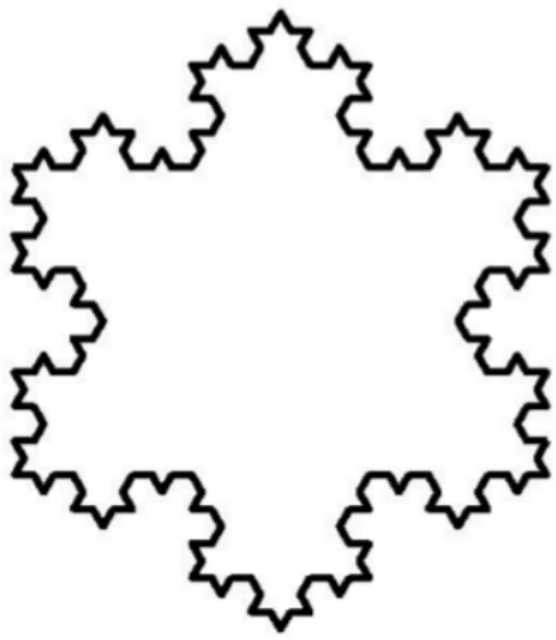
Problem 1 *Koch snowflake* (4 points)
In this exercise, we consider a sequence of subsets of $\mathbb{R}^2$ that is constructed in the following way: We start with an equilateral triangle A_0 with edge length 1. Now all straight lines are trisected into thirds and one replaces the middle third by a new (smaller) equilateral triangle. In this way, we obtain the figure A_1:

$A_1 =$ 

By doing this iteratively, we obtain a sequence of figures A_n for each $n \in \mathbb{N}$. To enable a better overview, you can see the next figures here:

$A_2 =$ 

$A_3 =$ 

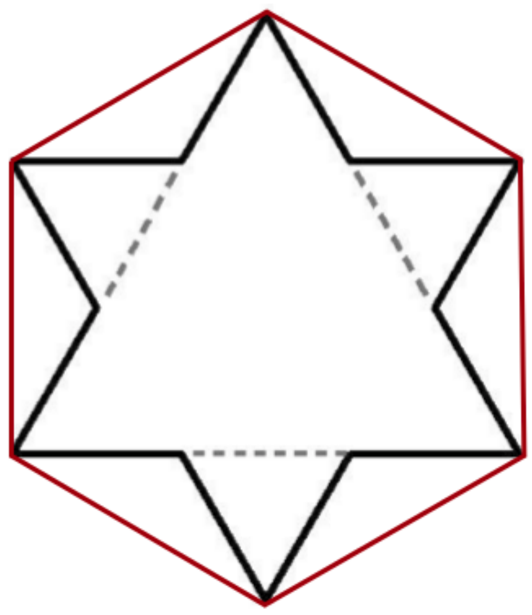
By defining in some new sense the limit of such a sequence $(A_n)_{n\in\mathbb{N}}$, we call this limit figure the *Koch snowflake*.

a) Let F_n be the surface area of A_n. This means that $(F_n)_{n\in\mathbb{N}}$ is a usual sequence of real numbers. Prove or disprove that the sequence $(F_n)_{n\in\mathbb{N}}$ is convergent.

b) Rewrite the sequence into a series by adding the new triangles in each step. Calculate then the area of the Koch snowflake.

Solutions:

(a) The sequence $(F_n)_{n\in\mathbb{N}}$ is bounded from above:



All triangles have to lie inside the hexagon.
(Can be proven with angles and triangles but this is not needed here in full detail)

Obviously, the sequence is monotonically increasing since we add new triangles in each step.

$\overset{\text{monotonicity criterion}}{\Longrightarrow}$ $(F_n)_{n\in\mathbb{N}}$ is convergent

(b) We define three sequences:

$e_k :=$ edge length of the <u>new</u> triangles in the n-th step

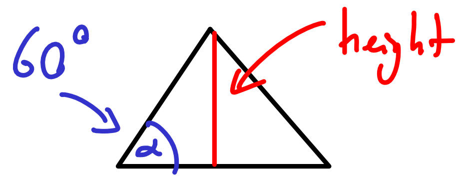
$h_k :=$ height of the <u>new</u> triangles in the n-th step

$d_k :=$ number of the <u>new</u> triangles in the n-th step

So we know: $e_1 = \frac{1}{3}$ for the first step.

New triangles are getting one third of the old edge length.

Therefore:
$$e_k = \left(\frac{1}{3}\right)^k$$

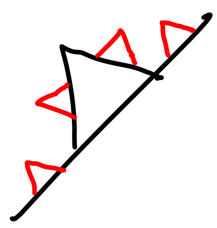


The height: $h_k = \tan(60°)\,\frac{e_k}{2} = \frac{\sqrt{3}}{2}\left(\frac{1}{3}\right)^k$

The number of new triangles:

$d_1 = 3, \quad d_2 = 3\cdot 4, \quad d_3 = 3\cdot 4\cdot 4$

Each edge gets 4 new triangles:



Hence: $d_k = 3\cdot 4^{k-1}$

The area is then:

$$F_n = F_0 + \sum_{k=1}^{n} d_k \cdot \frac{1}{2} e_k \cdot h_k$$

$$= \frac{\sqrt{3}}{4} + \sum_{k=1}^{n} 3\cdot 4^{k-1} \cdot \frac{1}{2}\left(\frac{1}{3}\right)^k \frac{\sqrt{3}}{2}\left(\frac{1}{3}\right)^k$$

$$= \frac{\sqrt{3}}{4} + \frac{3}{4}\cdot\frac{\sqrt{3}}{4}\cdot \underbrace{\sum_{k=1}^{n}\left(\frac{4}{9}\right)^k}_{\sum_{k=0}^{n} q^k - 1 \text{ with } q=\frac{4}{9}}$$

$$\Rightarrow \lim_{n\to\infty} F_n = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\cdot\left(\frac{1}{1-\frac{4}{9}} - 1\right) = \frac{2}{5}\sqrt{3}$$