Problem 3 *Majorant and minorant criterion* (4 points)
Use the majorant or minorant criterion from the tutorial to find out if the following series are convergent:

a) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2 + 3\sqrt{k}}$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(2k)}{k!}$

c) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{5k^2 + 7k^5}$

d) $\displaystyle\sum_{k=1}^{\infty} kq^k$ for $q \in [1, \infty)$.

Solutions

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2+3\sqrt{k}}$:

We have $\displaystyle\left|\frac{3}{k^2+\underbrace{3\sqrt{k}}_{\geq 0}}\right| \leq \frac{3}{k^2}$.

Since $\displaystyle\frac{3}{k^2} \geq 0$ and $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2} < \infty$, also

$\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^2+3\sqrt{k}} < \infty$ by the majorant criterion.

(b) $$\sum_{k=1}^{\infty} \frac{\sin(2k)}{k!}.$$

Since: $$\left|\frac{\sin(2k)}{k!}\right| = \frac{\overbrace{|\sin(2k)|}^{\leq 1}}{k!} \leq \frac{1}{k!}$$

Since $\frac{1}{k!} \geq 0$ and $\sum_{k=1}^{\infty} \frac{1}{k!} < \infty$ by the lecture,

also $\sum_{k=1}^{\infty} \frac{\sin(2k)}{k!} < \infty$ by the majorant criterion.

(c) $$\sum_{k=1}^{\infty} \frac{3}{5k^2+7k^5}.$$ We have $$\left|\frac{3}{5k^2+\underbrace{7k^5}_{\geq 0}}\right| \leq \frac{3}{5k^2}.$$

Since $\frac{3}{5k^2} \geq 0$ and $\sum_{k=1}^{\infty} \frac{3}{5k^2} < \infty$, also

$$\sum_{k=1}^{\infty} \frac{3}{5k^2+7k^5} < \infty$$ by majorant criterion.

(d) $$\sum_{k=1}^{\infty} k q^k$$ for $q \in [1, \infty)$.

Then: $k \cdot q^k \geq k \cdot 1 \geq k$

Since $\sum_{k=1}^{\infty} k = \infty$ and $k \geq 0$

the minorant criterion says that also

$$\sum_{k=1}^{\infty} k q^k = \infty$$