Problem 1 *Ratio and root test* (4 points)
Use the ratio or the root test to find if the following sequences converge absolutely:

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{n^{2n}}$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-7)^n}{4n}$

d) $\displaystyle\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$

Solutions

(a)

$$\sqrt[n]{\left|\frac{5n-3n^3}{7n^3+2}\right|^n} = \left|\frac{5\frac{1}{n^2}-3}{7+2\frac{1}{n^3}}\right| \xrightarrow{n\to\infty} \frac{3}{7} < 1$$

$\Longrightarrow$ absolutely convergent by root test

(b)

$$\frac{(2n+2)!}{(n+1)^{2n+2}}\cdot\frac{n^{2n}}{(2n)!} = \frac{(2n+2)(2n+1)}{n^2\left(1+\frac{1}{n}\right)^2}\cdot\frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^2}$$

$$= \frac{4+6\frac{1}{n}+2\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2}\cdot\frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^2} \xrightarrow{n\to\infty} \frac{4}{e^2} < 1$$

$\Longrightarrow$ absolutely convergent by ratio test

(c) $\displaystyle\frac{(-7)^n}{4n}$ does not converge to zero! $\Longrightarrow$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-7)^n}{4n}$ not abs. convergent!

(d)

$$\sqrt[n]{\frac{|(-10)^n|}{4^{2n+1}(n+1)}} = \frac{5}{8} \cdot \sqrt[n]{\frac{2^n}{2^n \cdot 4(n+1)}} = \frac{5}{8} \cdot \frac{1}{\sqrt[n]{(n+1) \cdot 4}}$$

$$\xrightarrow{n \to \infty} \frac{5}{8} < 1$$

$\Longrightarrow$ absolutely convergent by root test