Problem 2 *Convergence of series* (4 points)
Prove that the following series are convergent (by using the convergent criteria from Chapter 3).

a) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$, for $x \in (0, 2\pi)$

c) $\displaystyle\sum_{n=2}^{\infty} \frac{e^{i\pi(n+2n^2)}}{\ln(\ln n)}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{16}{8n^3 - 12n^2 - 2n + 3}$

Solutions

(a) $\cos(n\pi) = (-1)^n$ and $\ln : \mathbb{R} \to \mathbb{R}$ strictly monotonically increasing

with $\frac{1}{\ln(n)} \xrightarrow{n\to\infty} 0$

$\Longrightarrow \displaystyle\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln n}$ convergent by Leibniz criterion.

(b) We have $\frac{1}{n} \xrightarrow{n\to\infty} 0$ monotonically.

and: $$\sum_{n=1}^{N} \underbrace{e^{inx}}_{q^n} = \left| \frac{1 - e^{i(n+1)x}}{1 - \underbrace{e^{ix}}_{q}} - 1 \right| \leq \frac{2}{|1 - e^{ix}|} + 1$$

geometric sum formula

bounded, for $x \in (0, 2\pi)$

Theorem 2.23. Dirichlet criterion
If the series $\sum_{k=1}^{\infty} a_k$ in $\mathbb{F}$ is bounded and if the real sequence $(b_k)_{k\in\mathbb{N}}$ converges monotonically to zero, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

$\displaystyle\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$, for $x \in (0, 2\pi)$ $\Longrightarrow$ converges by Dirichlet criterion

(c) $e^{i\pi(n+2n^2)} = \underbrace{e^{i\pi n}}_{(-1)^n} \cdot \underbrace{e^{2i\pi n^2}}_{=1} = (-1)^n$

and $\ln(\ln(n))$ is monotonically increasing with $\dfrac{1}{\ln(\ln(n))} \xrightarrow{n\to\infty} 0$

$$\Longrightarrow \sum_{n=2}^{\infty} \frac{e^{i\pi(n+2n^2)}}{\ln(\ln n)}$$ is convergent by Leibniz criterion.

(d) Try majorant criterion: $\dfrac{16}{8n^3-12n^2-2n+3} \leq C \cdot \dfrac{1}{n^2}$ for a fixed $C>0$

$$\Longleftrightarrow \frac{16n^2}{8n^3-12n^2-2n+3} \leq C$$

$$\Longleftrightarrow \underbrace{\frac{16\frac{1}{n}}{8-12\frac{1}{n}-2\frac{1}{n^2}+3\frac{1}{n^3}}}_{\xrightarrow{n\to\infty}\ \frac{0}{8}=0} \leq C$$

Left-hand side is convergent to with limit 0. This means that there is a $N \in \mathbb{N}$ such that

$$\frac{16\frac{1}{n}}{8-12\frac{1}{n}-2\frac{1}{n^2}+3\frac{1}{n^3}} \leq \underbrace{1}_{\stackrel{!}{=}C} \quad \text{for all } n \geq N$$

$$\Longrightarrow \sum_{n=N}^{\infty} \frac{16}{8n^3-12n^2-2n+3}$$ convergent by majorant criterion

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{16}{8n^3-12n^2-2n+3}$$ also convergent (since we only add finitely many terms)