Problem 1 *Continuous functions* (4 points)
Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be two functions. Which of the following statements are true and which of them are (in general) false? Construct a counterexample for each false statement and give a proof for the correct ones.

a) If $fg : I \to \mathbb{R}$, $x \mapsto f(x) \cdot g(x)$ is continuous, then f and g are continuous.

b) If f and g are continuous functions, then $\max\{f, g\} : I \to \mathbb{R}$, $x \mapsto \max\{f(x), g(x)\}$ and $\min\{f, g\} : I \to \mathbb{R}$, $x \mapsto \min\{f(x), g(x)\}$ are continuous functions.
Hint: Try to rewrite the max *and* min *function using an absolute value.*

c) If f is injective and continuous, then $f^{-1} : f(I) \to I$ is continuous.
Hint: Who said that I has to be an interval?

d) If there exists an $L \geq 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in I$, then f is continuous.

Solutions

(a) Counterexample: $f(x) = 0$ for all $x \in \mathbb{R}$

$$g(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(b) For two real numbers $a, b \in \mathbb{R}$ one can show

$$2 \cdot \max\{a, b\} = a + b + |a - b|$$

If $a > b$: $a + b + |a - b| = 2a$ ✓
If $a < b$: $a + b + |a - b| = 2b$ ✓

Therefore the max-function is given by

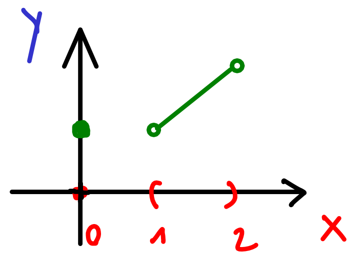
$$\max\{f(x), g(x)\} = \frac{1}{2}\big(f(x) + g(x) + |f(x) - g(x)|\big).$$

Since the absolute value is continuous, the combination above is also continuous.

Same idea for minimum-function:

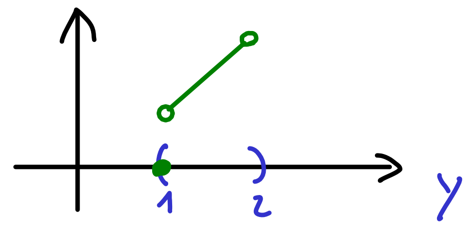
$$\min\{f(x), g(x)\} = \frac{1}{2}\big(f(x) + g(x) - |f(x) - g(x)|\big).$$

(c) Counterexample: Consider $I = \{0\} \cup (1,2)$



inverse function

(mirroring)



continuous! not continuous!

$f: I \to (1,2)$ $\qquad$ $f^{-1}: (1,2) \to I$

$f(x) = \begin{cases} 1, & x = 0 \\ x, & \text{else} \end{cases}$ $\qquad$ $f^{-1}(y) = \begin{cases} 0, & y = 1 \\ y, & \text{else} \end{cases}$

(d) Claim: $|f(x) - f(y)| \leq L\,|x-y| \Rightarrow f$ continuous

Proof: Let $x_0 \in I$ and $(x_n)_{n\in\mathbb{N}} \subseteq I$ a sequence with limit x_0 (for example: constant sequence $x_n = x_0$)

Let $\varepsilon > 0$ be arbitrary.

Then: $|f(x_0) - f(x_n)| \leq L\,|x_0 - x_n|$

for all $n \in \mathbb{N}$. Since the righthand side can be made arbitrarily small, we find an $N \in \mathbb{N}$ with

$$|x_0 - x_n| \leq \frac{\varepsilon}{L+1} \text{ for all } n \geq N.$$

Therefore:

$$|f(x_0) - f(x_n)| \leq \frac{L}{L+1} \cdot \varepsilon < \varepsilon$$

By definition: $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

Since this holds for all sequences $(x_n)_{n\in\mathbb{N}}$ with limit x_0, we have proven the continuity of $f: I \to \mathbb{R}$. $\square$

Remark: A function $f: I \to \mathbb{R}$ with the property: $\exists L \geq 0$ $|f(x) - f(y)| \leq L\,|x-y|$ for all $x,y \in I$ is called Lipschitz continuous.