Problem 2 *Limits of functions* (4 points)
Examine the following limits and evaluate them if they exist:

a) $\lim\limits_{x\to 1} \dfrac{x^5-1}{x^3-1}$

b) $\lim\limits_{x\to 0} \dfrac{x-|x|}{2x}$

c) $\lim\limits_{x\to 2} \left(\dfrac{2}{2x-x^2}+\dfrac{1}{x^2-3x+2}\right)$

d) $\lim\limits_{x\to \infty} \left(x-\sqrt{\dfrac{x^3+x}{x+1}}\right)$

Solutions: (a) Use polynomial division or Problem 2.2:

$$a^n-b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} \quad \text{for } a,b\in\mathbb{R},\ n\in\mathbb{N}.$$

Therefore: $x^5-1 = (x-1)\cdot \sum_{k=0}^{4} x^k$, $x^3-1 = (x-1)\cdot\sum_{k=0}^{2} x^k$

$$\Rightarrow \frac{x^5-1}{x^3-1} = \frac{x^4+x^3+x^2+x^1+x^0}{x^2+x^1+x^0} \xrightarrow{x\to 1} \underline{\frac{5}{3}}$$

(b) $$\frac{x-|x|}{2x} = \begin{cases} 0, & x>0 \\ 1, & x<0 \end{cases} \Rightarrow \text{limit } \underline{\text{doesn't exist}}.$$

(c) $$\frac{2}{2x-x^2}+\frac{1}{x^2-3x+2} = \frac{2}{x(2-x)}+\frac{1}{(x-2)(x-1)}$$

$$= \frac{-2(x-1)+x}{x(2-x)(x-1)} = \frac{-(x-2)}{x(2-x)(x-1)} = \frac{-1}{x(x-1)}$$

$$\xrightarrow{x\to 2} \underline{-\frac{1}{2}}$$

(d)

$$x - \sqrt{\frac{x^3+x}{x+1}} = \frac{x^2 - \frac{x^3+x}{x+1}}{x + \sqrt{\frac{x^3+x}{x+1}}} = \frac{(x^3+x^2) - x^3 - x}{(x+1)\left(x + \sqrt{\frac{x^3+x}{x+1}}\right)}$$

$$= \frac{x^2-x}{x+1} \cdot \frac{1}{x + \sqrt{x^2 \cdot \left(\frac{1+\frac{1}{x^2}}{1+\frac{1}{x}}\right)}} = \frac{\left(1-\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)} \cdot \frac{1}{1+\sqrt{\frac{1+\frac{1}{x^2}}{1+\frac{1}{x}}}}$$

$$\xrightarrow{x \to \infty} \frac{1}{1} \cdot \frac{1}{1+1} = \underline{\frac{1}{2}}$$