Problem 3 *Continuity* (4 points)
Are the following functions continuous on $\mathbb{R}$? For each case, give a proof of your conjecture.

a) $f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

c) $h(x) := \begin{cases} \frac{x^3+x^2-x-1}{x-1} & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$

b) $g(x) := \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

d) $j(x) := \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Solutions:

(a) Not continuous since the sequence $x_n = \dfrac{1}{2\pi n + \frac{\pi}{2}}$ converges to zero

but $f(x_n) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = \sin\left(\frac{\pi}{2}\right) = 1$

for all $n \in \mathbb{N}$ $\Rightarrow \lim\limits_{n\to\infty} f(x_n) \neq f(0)$

(b) g is continuous on $\mathbb{R}\setminus\{0\}$ since the product of two continuous functions is continuous.

For the origin we choose a null sequence $(x_n)_{n\in\mathbb{N}}$ and calculate: $|g(x_n)| \leq |x_n| \cdot \underbrace{|\sin(x_n)|}_{\leq 1} \xrightarrow{n\to\infty} 0$

Therefore, g is also continuous in 0.

(c) Polynomial division: $(x^3+x^2-x-1):(x-1) = x^2+2x+1$

Therefore h is continuous on $\mathbb{R}\setminus\{1\}$ as a rational funct.

and even at $x=1$ since $\lim\limits_{x\to 1} h(x) = \lim\limits_{x\to 1}(x^2+2x+1) = 4$

(d)

j is continuous on $\mathbb{R}\setminus\{0\}$ since the product of two continuous functions is continuous. ($\sin(x)$ cont. and $\frac{1}{x}$ cont.)

For showing continuity in $x = 0$:

Consider the series for $\sin(x)$ (see tutorial):

$$\frac{\sin(x)}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \cdot \frac{1}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$$

For an $x \in \mathbb{R}$ with $|x| < 1$ we have:

(setting $q := |x|^2$)

$$\left| \frac{\sin(x)}{x} - 1 \right| \leq \sum_{k=1}^{\infty} \frac{(x^2)^k}{(2k+1)!} \leq \sum_{k=1}^{\infty} q^k$$

$$\leq \sum_{k=0}^{\infty} q^k - 1 = \frac{1}{1-q} - 1$$

$$= \frac{q}{1-q} = \frac{|x|^2}{1-|x|^2}$$

Therefore, for a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to 0$, we have $\left| \frac{\sin(x_n)}{x_n} - 1 \right| \leq \frac{|x_n|^2}{1-|x_n|^2} \xrightarrow{n\to\infty} 0$.

This means that $j(x)$ is also continuous in $x = 0$.