Problem 1 *Pointwise and uniform convergence* (4 points)
We consider a sequence of functions $(f_n)_{n\in\mathbb{N}}$ given by $f_n : \mathbb{R} \to \mathbb{R}$ for all $n \in \mathbb{N}$.

a) Prove that $f_n(x) := 1 - \chi_{[-n,n]}(x)$ converges pointwisely, but not uniformly, to 0.

b) Prove that $f_n(x) := \big(1 - \chi_{[-n,n]}(x)\big)\, \mathrm{e}^{-|x|}$ converges uniformly to 0.

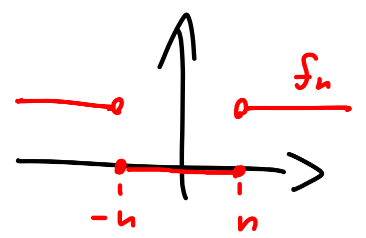
Here, the function $\chi_{[-n,n]} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi_{[-n,n]}(x) := \begin{cases} 1 & \text{if } x \in [-n,n], \\ 0 & \text{else.} \end{cases}$$

Solutions

(a) $|f_n(x)| = 0$ for all $n > |x|$

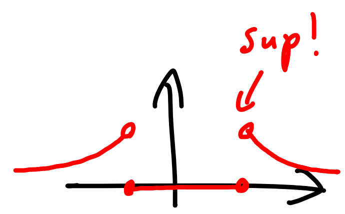
Therefore: $|f_n(x) - 0| \xrightarrow{n\to\infty} 0$



On the other hand: $\|f_n\|_\infty = \sup\limits_{x\in\mathbb{R}} |f_n(x)| = 1$ for all $n \in \mathbb{N}$

Therefore: $\|f_n - 0\|_\infty \overset{n\to\infty}{\not\longrightarrow} 0$.

(b) $\|f_n\|_\infty = \sup\limits_{x\in\mathbb{R}} |f_n(x)| = \sup\limits_{x\in\mathbb{R}} \left| (1 - \chi_{[-n,n]}(x))\, e^{-|x|} \right|$



$= 1 \cdot e^{-n} \xrightarrow{n\to\infty} 0$

Therefore: $\|f_n - 0\|_\infty$ converges to 0 $\Rightarrow$ uniform convergence.