Problem 3 *Existence of solutions* (4 points)
A function $p : \mathbb{R} \to \mathbb{R}$ of the form $p(x) = \sum_{k=0}^{n} a_k x^k$ with $a_k \in \mathbb{R}$ is called a *real polynomial of degree* n if $a_n \neq 0$. Prove the following:

a) If n is odd, then there is at least one $x_0 \in \mathbb{R}$ with $p(x_0) = 0$.
Remark: Such an x_0 is called a zero of the polynomial.

b) If n is even and additionally $a_n \cdot a_0 < 0$ holds, then there is at least one zero of p.

c) If n is even and additionally $p(\zeta) \cdot a_n < 0$ for a $\zeta \in \mathbb{R}$ holds, then p has at least two zeros.

Now consider the real exponential function $x \mapsto e^x$, which is strictly monotonically increasing.

d) Show that the equation $x = e^{-x}$ has exactly one real solution.

Solutions

(a) Since n is odd, we have the limits

$$\lim_{x \to \infty} p(x) = \operatorname{sgn}(a_n) \cdot \infty$$ and

$$\lim_{x \to -\infty} p(x) = -\operatorname{sgn}(a_n) \cdot \infty$$

Hence if we restrict the polynomial to a large enough number $N \in \mathbb{N}$,

$p : [-N, N] \to \mathbb{R}$ such that $p(N)$ and $p(-N)$ have diffent signs,

we can apply the intermediate theorem.

(b) Without loss of generality: $a_n > 0$, $a_0 < 0$

$\Rightarrow p(0) = a_0 < 0$

We have $\lim_{x \to \pm\infty} p(x) = \infty$, i.e $\exists \tilde{x} \in \mathbb{R}$ with $p(\tilde{x}) > 0$

Now we can also apply the intermediate theorem.

(c) Without loss of generality: $a_n > 0$, $p(\xi) < 0$.

We have $\lim_{x \to \pm\infty} (x) = \infty$, and therefore we find

x_1, x_2 with $x_1 < \xi < x_2$ with $p(x_1) > 0$

and $p(x_2) > 0$

Now apply the intermediate theorem for the restrictions

$p|_{[x_1, \xi]}$ and $p|_{[\xi, x_2]}$ and you will

find two diffent zeros.

(d) $e^{-x} - x$ is continous. $e^{-0} - 0 = 1 > 0$

$e^{-1} - 1 = \frac{1}{e} - 1 < 0$ $(e > 1)$

The intermediate theorem gives us a $y \in (0,1)$

with $e^{-y} - y = 0 \Rightarrow e^{-y} = y$

Since e^{-x} is injective, this is the only solution.